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Hilbert curves of 3-dimensional scrolls over surfaces

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ABSTRACT

Let (X, L) be a 3-dimensional scroll over a smooth surface Y. Its Hilbert curve is an affine plane cubic consisting of a given line and a conic. This conic turns out to be the Hilbert curve of the $\mathbb{Q}$-polarized surface $(Y, \frac{1}{2}\det\mathcal{E})$, where $\mathcal{E}$ is the rank-2 vector bundle obtained by pushing down L via the scroll projection, if and only if $\mathcal{E}$ is properly semistable in the sense of Bogomolov.

0. Introduction

The Hilbert curve of a polarized manifold was introduced in [5] and its study has been continued in [10,11,4]. The natural expectation is that several properties of the polarized manifold are encoded by this object. In fact a relevant property of the Hilbert curve is its sensitivity with respect to fibrations that suitable adjoint linear systems to the polarizing line bundle may induce on the manifold [5, Theorem 6.1]. The case of projective bundles over a smooth curve, with special emphasis on scrolls, has been widely discussed in [10]. Other examples with special regard to threefolds are presented in [5]. However, the case of scrolls over a surface is not yet discussed in the literature, not even for dimension three. Filling this gap is exactly the aim of this paper. Moreover, confining to threefolds we get a precise parallel with the case of quadric fibrations over a smooth curve studied in [5, Proposition 4.8]. Recall that these two types of varieties play a similar role in adjunction theory. In particular, in the setting we consider, a precise answer is given to a problem raised in [5].

Here is a summary of the content. Let (X, L) be a 3-dimensional scroll over a smooth surface Y, and let $\mathcal{E} = \pi_* L$, where $\pi : X \to Y$ is the scroll projection. According to [5, Theorem 6.1], the Hilbert curve $\Gamma_{(X,L)}$ of (X, L) is reducible into a given line ℓ and a conic, say G. In Section 2 we determine explicitly its canonical equation. The problem whether the resulting conic G itself can in turn be the Hilbert curve of any $\mathbb{Q}$-polarized surface seems not affordable in the general case, due to a too large number of variables. In

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fact, in Section 3, we present some elementary examples illustrating a range of possibilities. This suggests to confine the problem to the case where the underlying surface is the base itself, Y, of the scroll. In this context, the Hodge index theorem provides a necessary condition: an upper bound expressed in terms of K_Y and of the ample rank-2 vector bundle $\mathcal{E}$, that the Bogomolov number of $\mathcal{E}$ has to satisfy. On the other hand, the base surface Y is endowed with a natural polarization, namely $\det \mathcal{E}$. Addressing the specific question raised in [5, Problem 6.6 (2)], we can then ask whether the conic G is the Hilbert curve of Y with some $\mathbb{Q}$-polarization related to $\det \mathcal{E}$. What we prove in Section 4 is that G is the Hilbert curve of $(Y, \frac{1}{2}\det \mathcal{E})$ up to HC-equivalence (see [11]), if and only if $\mathcal{E}$ is properly semistable (in the sense of Bogomolov).

1. Background material

Varieties considered in this paper are defined over the field $\mathbb{C}$ of complex numbers. We use the standard notation and terminology from algebraic geometry. A manifold is any smooth projective variety; a surface is a manifold of dimension 2. The symbol $\equiv$ will denote numerical equivalence. With a little abuse, we adopt the additive notation for the tensor products of line bundles. The pullback of a vector bundle $\mathcal{F}$ on a manifold X by an embedding $Y \hookrightarrow X$ is simply denoted by $\mathcal{F}_Y$. We denote by T_X and K_X the tangent bundle and the canonical bundle of a manifold X, respectively. A polarized manifold is a pair (X, L) consisting of a manifold X and an ample line bundle L on X. The word scroll has to be intended in the classical sense. We denote by $\mathbb{F}_e := \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)\big)$ the Segre–Hirzebruch surface of invariant e ($e \geq 0$), and C_0 and f will stand for the tautological section and a fiber respectively, as in [8, p. 373]. Clearly, $(\mathbb{F}_0, [aC_0 + bf]) = \big(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)\big)$.

For the notion and the general properties of the Hilbert curve associated to a polarized manifold we refer to [5], see also [10]. Here we just recall some basic facts. Let (X, L) be a polarized manifold of dimension $n \geq 2$: if $\mathrm{rk}\langle K_X, L\rangle = 2$ we can consider $\mathrm{N}(X) := \mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ as a complex affine space and inside it the plane $\mathbb{A}^2 = \mathbb{C}\langle K_X, L\rangle$, generated by the classes of K_X and L. For any line bundle D on X the Riemann–Roch theorem provides an expression for the Euler–Poincaré characteristic $\chi(D)$ in terms of D and the Chern classes of X. Let p denote the complexified polynomial of $\chi(D)$, when we set $D = xK_X + yL$, with x, y complex numbers, namely $p(x, y) = \chi(xK_X + yL)$. The Hilbert curve of (X, L) is the complex affine plane curve $\Gamma = \Gamma_{(X,L)} \subset \mathbb{A}^2$ of degree n defined by $p(x, y) = 0$ [5, Section 2]. Taking into account that $c := \frac{1}{2}K_X$ is the fixed point of the Serre involution $D \mapsto K_X - D$ acting on $\mathrm{N}(X)$, it is convenient to represent Γ in terms of affine coordinates $(u = x - \frac{1}{2}, v = y)$ centered at c instead of (x, y). In other words, rewrite our divisor as $D = \frac{1}{2}K_X + E$, where $E = uK_X + vL$. Then Γ can be represented with respect to these coordinates by $p(\frac{1}{2} + u, v) = 0$. An obvious advantage is that, due to Serre duality, Γ is symmetric with respect to c (the origin in the (u, v)-plane). We refer to $p(\frac{1}{2} + u, v) = 0$ as the *canonical equation* of Γ. Another consequence of Serre duality is that $c \in \Gamma$ if n is odd, while if n is even and $\Gamma \ni c$, then c is a singular point of Γ [5, Section 2].

According to the above, $\chi(D)$ can be re-expressed in terms of E and the Chern classes of X in a nice way. In particular, for $n = 2$ we get

$$\chi(D) = \frac{1}{2}E^2 + \big(\chi(\mathcal{O}_X) - \frac{1}{8}K_X^2\big). \tag{1}$$

If $n = 3$, recalling that $\chi(\mathcal{O}_X) = -\frac{1}{24}K_X \cdot c_2$, where $c_2 = c_2(X)$, the usual expression of the Riemann–Roch theorem (e. g., see [8, p. 437]) takes the more convenient form

$$\chi(D) = \frac{1}{6}E^3 + \frac{1}{24}E \cdot (2c_2 - K_X^2). \tag{2}$$

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