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Monotonic linear recurrences

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Three-term (or, second order), linear, recurrence sequences can have magnitudes that grow or shrink monotonically. The rate of change can be tightly estimated a-priori. Applications are Bessel functions, eigenvectors of tridiagonal matrices, and the Lanczos method.

1. Introduction

Parlett [1, §2] in this journal recently discovered inequalities among consecutive entries in eigenvectors of tridiagonal matrices. His proof depends on triangular decomposition of diagonally dominant matrices in the form stated, for example, by Golub and Van Loan [2, §3.4.10].

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This paper recasts Parlett's inequalities: (i) with a stronger conclusion, (ii) a simpler proof, and (iii) in the more applicable setting of recurrence sequences. Briefly, if central coefficients of three-term recurrence formulas dominate the other coefficients in magnitude, for some range of consecutive indices $p \le k \le q$,

$$\begin{aligned} k^{\text{th}} \text{ recurrence formula:} &\quad a_{k-1}x_{k-1} + b_k x_k + c_{k+1}x_{k+1} = 0 && (1) \\ \textbf{dominant} \text{ central coefficient:} &\quad |b_k| > |a_{k-1}| + |c_{k+1}| && (2) \\ \text{nonzero peripheral coefficients:} &\quad \text{(i) } a_{k-1} \neq 0 \text{ and (ii) } c_{k+1} \neq 0 && (3) \\ \text{not all terms zero:} &\quad x_{k-1},\ x_k,\ x_{k+1} && (4) \end{aligned}$$

then the magnitudes, $|x_k|$ for $p-1 \le k \le q+1$, either are monotonic, or have exactly one inflection where the magnitudes switch from decreasing to increasing. Further, ratios of consecutive magnitudes satisfy a-priori bounds. The formula coefficients (1) and sequence terms (4) may be real numbers or complex numbers.

The monotonic property is demonstrated for Bessel functions, for Parlett's case of eigenvectors of tridiagonal matrices, and for the Lanczos algorithm.

Regularity of magnitudes in certain recurrence sequences is a basic fact of numerical mathematics that was overlooked in spite of many years studying recurrence formulas. For surveys of the historical theory see Lakshmikantham and Trigiante [3] and Milne-Thomson [4].

2. Criteria for monotonic sequences

Theorem 1 *(Growing sequence). If for all $p \le k \le q$*

(A) *conditions (1), (2), and (3.ii) hold*

and if at the ***start***

(B) $|x_{p-1}| \le |x_p|$
(C) $x_p \neq 0$

then the magnitudes $|x_p|, \ldots, |x_q|, |x_{q+1}|$ strictly ***increase*** *with ratios sandwiched below and above by the following inequalities for $p \le k \le q$,*

$$1 < \frac{|b_k| - |a_{k-1}|}{|c_{k+1}|} \le \frac{|x_{k+1}|}{|x_k|} \le \frac{|b_k| + |a_{k-1}|}{|c_{k+1}|} \ . \tag{5}$$

The numbers in these formulas may be real or complex.

Proof. The theorem follows by induction on k. In the initial case, $k = p$, assumptions (2) and (3.ii) are equivalent to the first inequality in (5). The second and third inequalities are