Linear Algebra and its Applications 537 (2018) 118–147

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

A generalized inverse for graphs with absorption

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ARTICLE INFO

Article history:
Received 11 October 2016
Accepted 26 September 2017
Available online 28 September 2017
Submitted by R. Brualdi

MSC:
05C
05C81
15A09
92B
94C

Keywords:
Networks
Random walk
Group inverse
Graph Laplacian

ABSTRACT

We consider weighted, directed graphs with a notion of absorption on the vertices, related to absorbing random walks on graphs. We define a generalized inverse of the graph Laplacian, called the absorption inverse, that reflects both the graph structure as well as the absorption rates on the vertices. Properties of this generalized inverse are presented, including a basic relationship between the absorption inverse and the group inverse of a related graph, a forest theorem for interpreting the entries of the absorption inverse, as well as relationships between the absorption inverse and the fundamental matrix of the absorbing random walk. Applications of the absorption inverse for describing the structure of graphs with absorption are given, including a directed distance metric, spectral partitioning algorithm, and centrality measure.

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https://doi.org/10.1016/j.laa.2017.09.029
0024-3795/

1. Introduction

This paper concerns a generalized inverse of the graph Laplacian that is related to transient random walks on graphs. Let G denote a strongly connected, weighted, directed graph on the vertex set $V = \{1, 2, \ldots, n\}$. Let $A \in \mathbb{R}^{n\times n}$ be the weighted adjacency matrix with ijth entry a_{ij} corresponding to the weight of the edge from j to i. The outdegree (i.e. sum of outgoing weights) from vertex i is given by $w_i = \sum_{j=1}^{n} a_{ji}$. The graph Laplacian matrix is given by $L = W - A$ where $W = \mathrm{diag}\{w\}$, the diagonal matrix with $W_{ii} = w_i$. Both the group inverse, $L^{\#}$, and the Moore–Penrose pseudoinverse, $L^{\dagger}$, of the graph Laplacian have been shown to encode valuable information about the graph as well as the regular Markov process generated by L including hitting times, measures of centrality, distance functions on trees, and more [1–5].

Here we study a generalization of the group inverse that similarly captures valuable information about transient random walks on the graph, where each vertex is a transient state with transition rate $d_i > 0$ to an absorbing state. We will refer to the d_i as *absorption rates*. Let $d = (d_1, \ldots, d_n)^T$ denote the (column) vector of absorption rates, and let $D = \mathrm{diag}\{d\}$. Formally, we define a *graph with absorption* as the pair (G, d).

The object of interest for this paper is the matrix X which satisfies the two conditions listed below:

$$XLy = y \text{ for } y \in N_{1,0}, \tag{1}$$

$$Xy = 0 \text{ for } y \in R_{1,0}, \tag{2}$$

where

$$N_{1,0} = \{x \in \mathbb{R}^n : Dx \in \mathrm{range}\, L\}, \tag{3}$$

$$R_{1,0} = \{Dx : x \in \ker L\}. \tag{4}$$

That is, X sends $R_{1,0}$ to 0 and acts as the left inverse of L on $N_{1,0}$.

Definition 1. Let L be the graph Laplacian of a strongly connected graph with absorption (G, d), with $d > 0$. A matrix X satisfying (1)–(2) is said to be an absorption inverse of L with respect to d.

We will denote the absorption inverse of L with respect to d by L^d. As the name implies, L^d is indeed a generalized inverse of L. Specifically, Tien et al. [6] show that L^d is a $\{1, 2\}$-inverse of L. Derivation of L^d in [6] is through a Laurent series expansion, using the elegant results of Langenhop [7]. Indeed, Langenhop shows that conditions (1) and (2) arise naturally in the context of Laurent series expansion for the inverse of a nearly singular matrix. This motivates our definition of L^d.

Our interest in studying L^d is due to its connections with transient random walks on graphs, and in particular with how the absorption rates d_i influence how we think of