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Arithmetical structures on graphs ☆

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Arithmetical structures on a graph were introduced by Lorenzini in [9] as some intersection matrices that arise in the study of degenerating curves in algebraic geometry. In this article we study these arithmetical structures, in particular we are interested in the arithmetical structures on complete graphs, paths, and cycles. We begin by looking at the arithmetical structures on a multidigraph from the general perspective of M-matrices. As an application, we recover the result of Lorenzini about the finiteness of the number of arithmetical structures on a graph. We give a description on the arithmetical structures on the graph obtained by merging and splitting a vertex of a graph in terms of its arithmetical structures. On the other hand, we give a description of the arithmetical structures on the clique–star transform of a graph, which generalizes the subdivision of a graph. As an application of this result we obtain an explicit description of all the arithmetical structures on the paths and cycles and we show that the number of the arithmetical structures on a path is a Catalan number.

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1. Introduction

Given a loopless multidigraph $G = (V, E)$, its *generalized Laplacian matrix* is given by

$$L(G, X_G)_{u,v} = \begin{cases} -m_{u,v} & \text{if } u \neq v, \\ x_u & \text{if } u = v, \end{cases}$$

where $m_{u,v}$ is the number of arcs between u and v and $X_G = \{x_v | v \in V\}$ is a finite set of variables indexed by the vertices of G. This generalized Laplacian matrix was introduced in [4] and is very similar to the corresponding concept introduced by Godsil and Royle in [8, Section 13.9]. For any $\mathbf{d} \in \mathbb{Z}^V$, let $L(G, \mathbf{d})$ be the integral matrix that results by making $x_u = \mathbf{d}_u$ on $L(G, X_G)$, which is called the Laplacian matrix of G and $(\mathbf{d}, \mathbf{r})$. Clearly, the *adjacency matrix* of G is equal to $-L(G, \mathbf{0})$ and their *Laplacian matrix* is equal to $L(G, \mathbf{deg}_G)$, where $\mathbf{deg}_G$ is the out-degree vector of G. The Laplacian matrix of a graph is very important in spectral graph theory and in general in algebraic graph theory, see for instance [8] and the references therein. The Laplacian matrices with which we work here essentially correspond to integral matrices with non-positive entries off the diagonal.

Some combinatorial properties of a multidigraph G are coded in its Laplacian matrix. For instance, G is said to be *strongly connected* if for any two vertices $u, v \in V$ there exists a directed path from u to v. It is well known that G is strongly connected if and only if $L(G, \mathbf{deg}_G)$ is an irreducible matrix, see for instance [8]. Recall that a square matrix A is called reducible if there exists a permutation matrix P such that

$$PAP^t = \begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix}$$

for some non-trivial square matrices A_1 and A_2. Moreover, is not difficult to check that G is strongly connected if and only if $L(G, \mathbf{d})$ is an irreducible matrix for any vector $\mathbf{d}$ in $\mathbb{Z}^V$.

Now, we define the main object of this article. An *arithmetical graph* is a triplet $(G, \mathbf{d}, \mathbf{r})$ given by a multidigraph G and a pair of vectors $(\mathbf{d}, \mathbf{r}) \in \mathbb{N}_+^V \times \mathbb{N}_+^V$ such that $\gcd(\mathbf{r}_v \,|\, v \in V(G)) = 1$ and

$$L(G, \mathbf{d})\mathbf{r}^t = \mathbf{0}^t.$$

Note that we impose the condition that all the entries of $\mathbf{d}$ and $\mathbf{r}$ are positive. Given an arithmetical graph $(G, \mathbf{d}, \mathbf{r})$, we say that the pair $(\mathbf{d}, \mathbf{r})$ is an *arithmetical structure* on G. This concept was introduced by Lorenzini in [9] as some intersection matrices that arise in the study of degenerating curves in algebraic geometry, see for instance [10] for a geometric point of view. Any simple graph H has an arithmetical structure, given by