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Stability results for Gabor frames and the p-order hold models

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ABSTRACT

We prove stability results for a class of Gabor frames in $L^2(\mathbb{R})$. We consider window functions in the Sobolev spaces $H_0^1(\mathbb{R})$ and B-splines of order $p \geq 1$. Our results can be used to describe the effect of the timing jitters in the p-order hold models of signal reconstruction.

1. Introduction

In this paper we prove stability results for Gabor frames and bases of $L^2(\mathbb{R})$ that are relevant in electronics and communication theory. A *Gabor system* in $L^2(\mathbb{R})$ is a

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collection of functions $\mathcal{G} = \{e^{2\pi i b_n x} g(x - a_k)\}_{n,k\in\mathbb{Z}}$, where g (the *window function*) is a fixed function in $L^2(\mathbb{R})$ and $a_k,\ b_n \in \mathbb{R}$.

If $\mathcal{G}$ is *regular*, i.e., if $(a_k, b_n) = (ak, bn)$ for some $a, b > 0$, we let $\mathcal{G}(g, a, b) = \{e^{2\pi binx} g(x - ak)\}_{n,k\in\mathbb{Z}}$.

Gabor systems have had a fundamental impact on the development of modern time-frequency analysis and have been widely used in all branches of pure and applied sciences.

An important problem is to determine general and verifiable conditions on the window function g, the *time sampling* $\{a_k\}$ and the *frequency sampling* $\{b_n\}$ which imply that a Gabor system is a frame. In the regular case many necessary and sufficient conditions on g, a and b are known (see e.g. [1] and the references cited there). An early article by Gröchenig [2] provided some partial sufficient conditions for the existence of irregular Gabor frames. See also [3] and [4] and the articles cited in these papers.

Given a regular Gabor frame $\mathcal{G}(g, a, b)$, it is important to determine *stability bounds* $\delta_{n,k} > 0$ so that each set $\mathcal{F} = \{e^{2\pi i b\lambda_{n,k} x} g(x - a\mu_{n,k})\}_{n,k\in\mathbb{Z}}$ is a frame whenever $|\lambda_{n,k} - n| + |\mu_{n,k} - k| < \delta_{n,k}$. The main results of our paper concern the stability of Gabor frames $\mathcal{G}(\text{rect}^{(p)}, a, b)$ where $\text{rect}(x) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(x)$ is the characteristic function of the interval $[-\frac{1}{2}, \frac{1}{2}]$ and $\text{rect}^{(p)}(x) = \text{rect} * ... * \text{rect}(x)$ is the p-times iterated convolution of $\text{rect}(x)$. The function $\text{rect}^{(p)}(x)$ is a piecewise polynomial function of degree $p - 1$ and a prime example of *B-spline* of order $p - 1$. See [5], [6], [7] and the references cited there.

Our investigation is motivated by the study of the timing jitter effect in p-order hold (pOH) devices. The pOH devices are used to transform a sequence of impulses $\{q_n\}$ originating from a continuous-time signal $f(t)$ into a piecewise polynomial function $f_p(t)$. The impulses are assumed to be evenly spaced, i.e., $q_n = f(Tn)$ for some $T > 0$, but in the presence of timing jitter we have instead $q_n = f(T(n \pm \epsilon_n))$ for some $\epsilon_n > 0$. It is natural to investigate whether $f(t)$ can be effectively reconstructed from the sequence $f(T(n \pm \epsilon_n))$.

It is proved [8], (but see also [9, chapt. 11]) that the condition $0 < ab \le 1$ is necessary for a Gabor system $\mathcal{G}(g, a, b)$ to be a frame, so we will always assume (often without saying) that $ab \le 1$. When $\mathcal{G}(g, a, b)$ is a *Riesz basis*, i.e., it is the image of an orthonormal basis of $L^2(\mathbb{R})$ through a linear, invertible and bounded transformation, we have $ab = 1$.

We consider sets of coefficients $\{\mu_{n,k}\}_{n,k\in\mathbb{Z}} \subset \mathbb{R}$, with

$$\boxed{L_n = \sup_{k\in\mathbb{Z}} |\mu_{n,k} - k| < 1; \quad L = \sum_{n\in\mathbb{Z}} L_n < \infty.} \tag{1}$$

The assumption $L_n < 1$ is made to simplify the statement of our result, but it is not necessary in the proofs.

We prove first a stability result for the frame $\mathcal{G}(\text{rect}, a, b)$.

Theorem 1.1. *Assume* $0 < a \le 1$, *and* $4abL < 1$. *The set*

$$\mathcal{F} = \left\{e^{2\pi ibnt}\, \text{rect}\,(a\mu_{n,k} - t)\right\}_{k,n\in\mathbb{Z}}$$