Linear Algebra and its Applications 536 (2018) 201–209

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Wildness of the problems of classifying two-dimensional spaces of commuting linear operators and certain Lie algebras

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ARTICLE INFO

Article history:
Received 6 August 2017
Accepted 18 September 2017
Available online 21 September 2017
Submitted by P. Semrl

MSC:
15A21
16G60
17B10

Keywords:
Spaces of commuting linear operators
Matrix Lie algebras
Wild problems

ABSTRACT

For each two-dimensional vector space V of commuting $n \times n$ matrices over a field $\mathbb{F}$ with at least 3 elements, we denote by $\widetilde{V}$ the vector space of all $(n+1) \times (n+1)$ matrices of the form $\begin{bmatrix} A & * \\ 0 & 0 \end{bmatrix}$ with $A \in V$. We prove the wildness of the problem of classifying Lie algebras $\widetilde{V}$ with the bracket operation $[u, v] := uv - vu$. We also prove the wildness of the problem of classifying two-dimensional vector spaces consisting of commuting linear operators on a vector space over a field.

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https://doi.org/10.1016/j.laa.2017.09.019

1. Introduction

Let $\mathbb{F}$ be a field that is not the field with 2 elements. We prove the wildness of the problems of classifying

- two-dimensional vector spaces consisting of commuting linear operators on a vector space over $\mathbb{F}$ (see Section 2), and
- Lie algebras $\mathrm{L}(V)$ with bracket $[u, v] := uv - vu$ of matrices of the form

$$\begin{bmatrix} & & & \alpha_1 \\ & A & & \vdots \\ & & & \alpha_n \\ 0 & \dots & 0 & 0 \end{bmatrix}, \qquad \text{in which } A \in V, \quad \alpha_1, \dots, \alpha_n \in \mathbb{F}, \tag{1}$$

  in which V is any two-dimensional vector space of $n \times n$ commuting matrices over $\mathbb{F}$ (see Section 3).

A classification problem is called *wild* if it contains the problem of classifying pairs of $n \times n$ matrices up to similarity transformations

$$(M, N) \mapsto S^{-1}(M, N)S := (S^{-1}MS, S^{-1}NS)$$

with nonsingular S. This notion was introduced by Donovan and Freislich [8,9]. Each wild problem is considered as hopeless since it contains the problem of classifying an arbitrary system of linear mappings, that is, representations of an arbitrary quiver (see [13,5]).

Let $\mathcal{U}$ be an n-dimensional vector space over $\mathbb{F}$. The problem of classifying linear operators $\mathcal{A} : \mathcal{U} \to \mathcal{U}$ is the problem of classifying matrices $A \in \mathbb{F}^{n\times n}$ up to similarity transformations $A \mapsto S^{-1}AS$ with nonsingular $S \in \mathbb{F}^{n\times n}$. In the same way, the problem of classifying vector spaces $\mathcal{V}$ of linear operators on $\mathcal{U}$ is the problem of classifying matrix vector spaces $V \subset \mathbb{F}^{n\times n}$ up to similarity transformations

$$V \mapsto S^{-1}VS := \{S^{-1}AS \mid A \in V\} \tag{2}$$

with nonsingular $S \in \mathbb{F}^{n\times n}$ (the spaces V and $S^{-1}VS$ are *matrix isomorphic*; see [14]). In Theorem 1(a), we prove the wildness of the problem of classifying two-dimensional vector spaces $V \subset \mathbb{F}^{n\times n}$ of commuting matrices up to transformations (2).

Each two-dimensional vector space $V \subset \mathbb{F}^{n\times n}$ is given by its basis $A, B \in V$ that is determined up to transformations $(A, B) \mapsto (\alpha A + \beta B, \gamma A + \delta B)$, in which $\left[\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right] \in \mathbb{F}^{2\times 2}$ is a change-of-basis matrix. Thus, the problem of classifying two-dimensional vector spaces $V \subset \mathbb{F}^{n\times n}$ up to transformations (2) is the problem of classifying pairs of linear independent matrices $A, B \in \mathbb{F}^{n\times n}$ up to transformations

$$(A, B) \mapsto (A', B') := S^{-1}(\alpha A + \beta B, \gamma A + \delta B)S, \tag{3}$$