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# Low rank tensor recovery via iterative hard thresholding

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ABSTRACT

We study extensions of compressive sensing and low rank matrix recovery (matrix completion) to the recovery of low rank tensors of higher order from a small number of linear measurements. While the theoretical understanding of low rank matrix recovery is already well-developed, only few contributions on the low rank tensor recovery problem are available so far. In this paper, we introduce versions of the iterative hard thresholding algorithm for several tensor decompositions, namely the higher order singular value decomposition (HOSVD), the tensor train format (TT), and the general hierarchical Tucker decomposition (HT). We provide a partial convergence result for these algorithms which is based on a variant of the restricted isometry property of the measurement operator adapted to the tensor decomposition at hand that induces a corresponding notion of tensor rank. We show that subgaussian measurement ensembles satisfy the tensor restricted isometry property with high probability under a certain almost optimal bound on the number of measurements which depends on the corresponding tensor format. These bounds are extended to partial Fourier maps combined with random sign flips of the tensor entries. Finally, we illustrate the performance of iterative hard thresholding methods for tensor recovery via numerical experiments where

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we consider recovery from Gaussian random measurements, tensor completion (recovery of missing entries), and Fourier measurements for third order tensors.



## 1. Introduction and motivation

Low rank recovery builds on ideas from the theory of compressive sensing which predicts that sparse vectors can be recovered from incomplete measurements via efficient algorithms including $\ell_1$-minimization. The goal of low rank matrix recovery is to reconstruct an unknown matrix $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$ from linear measurements $\mathbf{y} = \mathcal{A}(\mathbf{X})$, where $\mathcal{A} : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^m$ with $m \ll n_1 n_2$. Since this is impossible without additional assumptions, one requires that $\mathbf{X}$ has rank at most $r \ll \min\{n_1, n_2\}$, or can at least be approximated well by a rank-$r$ matrix. This setup appears in a number of applications including signal processing [2,36], quantum state tomography [23,24,34,39] and recommender system design [10,11].

Unfortunately, the natural approach of finding the solution of the optimization problem

$$\min_{\mathbf{Z} \in \mathbb{R}^{n_1 \times n_2}} \operatorname{rank}(\mathbf{Z}) \quad \text{s.t.} \quad \mathcal{A}(\mathbf{Z}) = \mathbf{y}, \tag{1}$$

is NP hard in general. Nevertheless, it has been shown that solving the convex optimization problem

$$\min_{\mathbf{Z} \in \mathbb{R}^{n_1 \times n_2}} \|\mathbf{Z}\|_* \quad \text{s.t.} \quad \mathcal{A}(\mathbf{Z}) = \mathbf{y}, \tag{2}$$

where $\|\mathbf{Z}\|_* = \operatorname{tr}\left( (\mathbf{Z}^* \mathbf{Z})^{1/2} \right)$ denotes the nuclear norm of a matrix $\mathbf{Z}$, reconstructs $\mathbf{X}$ exactly under suitable conditions on $\mathcal{A}$ [10,18,36,50]. Provably optimal measurement maps can be constructed using randomness. For a (sub-)Gaussian random measurement map, $m \geq Cr \max\{n_1, n_2\}$ measurements are sufficient to ensure stable and robust recovery via nuclear norm minimization [9,36,50] and other algorithms such as iterative hard thresholding [60]. We refer to [34] for extensions to ensembles with four finite moments.

In this note, we go one step further and consider the recovery of low rank tensors $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$ of order $d \geq 3$ from a small number of linear measurements $\mathbf{y} = \mathcal{A}(\mathbf{X})$, where $\mathcal{A} : \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d} \to \mathbb{R}^m$, $m \ll n_1 n_2 \cdots n_d$. Tensors of low rank appear in a variety of applications such as video processing ($d = 3$) [40], time-dependent 3D imaging ($d = 4$), ray tracing where the material dependent bidirectional reflection function is an order four tensor that has to be determined from measurements [40], numerical solution of the electronic Schrödinger equation ($d = 3N$, where $N$ is the number of particles) [4,41,67], machine learning [51] and more.