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Unique determination of a time-dependent potential for wave equations from partial data

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Abstract

We consider the inverse problem of determining a time-dependent potential q, appearing in the wave equation $\partial_t^2 u - \Delta_x u + q(t,x)u = 0$ in $Q = (0,T) \times \Omega$ with $T > 0$ and Ω a $\mathcal{C}^2$ bounded domain of $\mathbb{R}^n$, $n \geqslant 2$, from partial observations of the solutions on ∂Q. More precisely, we look for observations on ∂Q that allows to recover uniquely a general time-dependent potential q without involving an important set of data. We prove global unique determination of $q \in L^\infty(Q)$ from partial observations on ∂Q. Besides being nonlinear, this problem is related to the inverse problem of determining a semilinear term appearing in a nonlinear hyperbolic equation from boundary measurements.

MSC: 35R30; 35L05

Keywords: Inverse problems; Wave equation; Time-dependent potential; Uniqueness; Carleman estimates; Partial data

1. Introduction

1.1. Statement of the problem

We fix Ω a $\mathcal{C}^2$ bounded domain of $\mathbb{R}^n$, $n \geqslant 2$, and we set $\Sigma = (0,T) \times \partial\Omega$, $Q = (0,T) \times \Omega$ with $0 < T < \infty$. We consider the wave equation

$$\partial_t^2 u - \Delta_x u + q(t,x)u = 0, \quad (t,x) \in Q, \tag{1.1}$$

where $q \in L^\infty(Q)$ is real valued. We study the inverse problem of determining q from observations of solutions of (1.1) on $\partial Q = \Sigma \cup (\{0\} \times \overline{\Omega}) \cup (\{T\} \times \overline{\Omega})$.

It is well known that for $T > \mathrm{Diam}(\Omega)$ the data

$$\mathcal{A}_q = \{(u_{|\Sigma}, \partial_\nu u_{|\Sigma}) : \ u \in L^2(Q),\ \Box u + qu = 0,\ u_{|t=0} = \partial_t u_{|t=0} = 0\} \tag{1.2}$$

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determines uniquely a time-independent potential q (e.g. [27]). Here ν denotes the outward unit normal vector to $\partial\Omega$, $\partial_\nu = \nu \cdot \nabla_x$ the normal derivative and from now on $\Box$ denotes the differential operator $\partial_t^2 - \Delta_x$. It has been even proved that partial knowledge of $\mathcal{A}_q$ determines a time-independent potential q (e.g. [9]). In contrast to time-independent potentials, we can not recover the restriction of a general time-dependent potential q to the set

$$D = \{(t,x) \in Q : \ 0 < t < \mathrm{Diam}(\Omega)/2, \ \mathrm{dist}(x, \partial\Omega) > t\}$$

from the data $\mathcal{A}_q$. Indeed, assume that $\Omega = \{x \in \mathbb{R}^n : \ |x| < R\}$, $T > R > 0$. Now let u solve

$$\Box u = 0, \ u_{|\Sigma} = f, \quad u_{|t=0} = \partial_t u_{|t=0} = 0$$

with $f \in H^1(\Sigma)$ satisfying $f_{|t=0} = 0$. Since $u_{|t=0} = \partial_t u_{|t=0} = 0$, the finite speed of propagation implies that $u_{|D} = 0$. Therefore, for any $q \in \mathcal{C}_0^\infty(D)$, we have $qu = 0$ and u solves

$$\Box u + qu = 0, \ u_{|\Sigma} = f, \quad u_{|t=0} = \partial_t u_{|t=0} = 0.$$

This last result implies that for any $q \in \mathcal{C}_0^\infty(D)$ we have $\mathcal{A}_q = \mathcal{A}_0$ where $\mathcal{A}_0$ stands for $\mathcal{A}_q$ when $q = 0$.

Facing this obstruction to uniqueness, it appears that four different approaches have been considered so far to solve this problem:

1) Considering the equation (1.1) for any time $t \in \mathbb{R}$ instead of $0 < t < T$ (e.g. [28,29]).
2) Recovering the restriction on a subset of Q of a time-dependent potential q from the data $\mathcal{A}_q$ (e.g. [26]).
3) Recovering a time-dependent potential q from the extended data C_q (e.g. [13]) given by

$$C_q = \{(u_{|\Sigma}, u_{|t=0}, \partial_t u_{|t=0}, \partial_\nu u_{|\Sigma}, u_{|t=T}, \partial_t u_{|t=T}) : \ u \in L^2(Q), \ (\partial_t^2 - \Delta_x + q)u = 0\}. \tag{1.3}$$

4) Recovering time-dependent coefficients that are analytic with respect to the t variable (e.g. [10]).

Therefore, it seems that the only results of unique global determination of a time-dependent potential q proved so far (at finite time) involve strong smoothness assumptions such as analyticity with respect to the t variable or the important set of data C_q. In the present paper we investigate some conditions that guaranty unique determination of general time-dependent potentials without involving an important set of data. More precisely, our goal is to prove unique global determination of a general time-dependent potential q from partial knowledge of the set of data C_q.

1.2. Physical and mathematical interest

Physically speaking, our inverse problem can be stated as the determination of physical properties such as the time evolving density of an inhomogeneous medium by probing it with disturbances generated on some parts of the boundary and at initial time. The data is the response of the medium to these disturbances, measured on some parts of the boundary and at the end of the experiment, and the purpose is to recover the function q which measures the property of the medium. Note also that the determination of time-dependent potentials can be associated with models where it is necessary to take into account the evolution in time of the perturbation.

We also precise that the determination of time-dependent potentials can be an important tool for the more difficult problem of determining a non-linear term appearing in a nonlinear wave equation from observations of the solutions on ∂Q. Indeed, in [15] Isakov applied such results for the determination of a semilinear term appearing in a semilinear parabolic equation from observations of the solutions on ∂Q.

1.3. Existing papers

In recent years the determination of coefficients for hyperbolic equations from boundary measurements has been growing in interest. Many authors have considered this problem with observations given by the set $\mathcal{A}_q$ defined by (1.2). In [27], Rakesh and Symes proved that $\mathcal{A}_q$ determines uniquely a time-independent potential q and [14] proved unique determination of a potential and a damping coefficient. The uniqueness by partial boundary observations has been considered in [9]. For sake of completeness we also mention that the stability issue related to this problem has been treated in [2,16,19,24,31,32]. Note that [19] extended the results of [27] to time-independent coefficients of order