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Localisation of directional scale-discretised wavelets on the sphere

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ABSTRACT

Scale-discretised wavelets yield a directional wavelet framework on the sphere where a signal can be probed not only in scale and position but also in orientation. Furthermore, a signal can be synthesised from its wavelet coefficients exactly, in theory and practice (to machine precision). Scale-discretised wavelets are closely related to spherical needlets (both were developed independently at about the same time) but relax the axisymmetric property of needlets so that directional signal content can be probed. Needlets have been shown to satisfy important quasi-exponential localisation and asymptotic uncorrelation properties. We show that these properties also hold for directional scale-discretised wavelets on the sphere and derive similar localisation and uncorrelation bounds in both the scalar and spin settings. Scale-discretised wavelets can thus be considered as directional needlets.

1. Introduction

Wavelet methodologies on the sphere are not only of considerable theoretical interest in their own right but also have important practical application. For example, wavelets analyses on the sphere have led to many insightful scientific studies in the fields of planetary science (*e.g.* [4,5]), geophysics (*e.g.* [13,34,68,69]) and cosmology, in particular for the analysis of the cosmic microwave background (CMB) (*e.g.* [7,9,15,30,40–43, 49,53,59–63,66,75–77,82,83]) (for a somewhat dated review see [50]), among others. Of particular importance in such applications is the scale-space trade-off of the wavelets adopted, which arises from the extension of the familiar (Euclidean) Fourier uncertainly principle to the sphere [56]. Consequently, characterising the localisation properties of wavelets on the sphere is of considerable interest.

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Many early attempts to extend wavelet transforms to the sphere differ primarily in the manner in which dilations are defined on the sphere [2,3,14,19,29,39,56,64,65,73]. The construction of Freeden & Windheuser [19] is based on singular integrals on the sphere, while Antoine and Vandergheynst [2,3] follow a group theoretic approach. In the latter construction dilation is defined via the stereographic projection of the sphere to the plane, leading to a consistent framework that reduces locally to the usual continuous wavelet transform in the Euclidean limit. An implementation and technique to approximate functions on the sphere has been developed for this approach [1]. This construction is revisited in [78], independently of the original group theoretic formalism, and fast algorithms are developed in [79,80].

Initial wavelet constructions were essentially based on continuous methodologies, which, although insightful, limited practical application to problems where the exact synthesis of a function from its wavelet coefficients is not required. Early discrete constructions [7,67,72] (and subsequently [47,52]) that support exact synthesis were built on particular pixelisations of the sphere and do not necessarily lead to stable bases [72]. Half-continuous and fully discrete frames based on the continuous framework of [2,3] were constructed by [10,11] and polynomial frames were constructed by [54]. More recently, a number of exact discrete wavelet frameworks on the sphere have been developed, with underlying continuous representations and fast implementations that have been made available publicly, including: needlets [6,36,55]; directional scale-discretised wavelets [33,48,81]; and the isotropic undecimated and pyramidal wavelet transforms [71]. Each approach has also been extended to analyse spin functions on the sphere [21–24,37,46,70] and functions defined on the three-dimensional ball formed by augmenting the sphere with the radial line [17,31,32,45].

1.1. Contribution

Needlets [6,36,55] and directional scale-discretised wavelets [33,48,81] on the sphere were developed independently, about the same time, but share many similarities. Both are essentially constructed by a Meyer-type tiling of the line defined by spherical harmonic degree ℓ. Directional scale-discretised wavelets in addition include a directional component in the wavelet kernel, yielding a directional wavelet analysis so that signal content can be probed not only in scale and position but also in orientation. Needlets have been shown to satisfy important quasi-exponential localisation and asymptotic uncorrelation properties [6,21,23,24,36,55,58]. In this article we show that these properties also hold for directional scale-discretised wavelets. We derive equivalent localisation and uncorrelation bounds, in both the scalar and spin settings, and show that directional scale-discretised wavelets are characterised by excellent localisation properties in the spatial domain.

More precisely, we prove that for any $\xi \in \mathbb{R}_*^+$, there exist strictly positive constants $C_1, C_2 \in \mathbb{R}_*^+$, such that the directional scale-discretised wavelet $\Psi \in \mathrm{L}^2(\mathbb{S}^2)$, defined on the sphere $\mathbb{S}^2$ and centred on the North pole, satisfies the localisation bound:

$$\big|\Psi(\theta,\varphi)\big| \leq \frac{C_1}{\big(1+C_2\theta\big)^{\xi}} \,, \tag{1}$$

where $(\theta,\varphi) \in \mathbb{S}^2$ denote spherical coordinates, with colatitude $\theta \in [0,\pi]$ and longitude $\varphi \in [0,2\pi)$. Furthermore, we prove that for Gaussian random fields on the sphere, directional scale-discretised wavelet coefficients are asymptotically uncorrelated. The correlation of wavelet coefficients corresponding to wavelets at scales $j, j' \in \mathbb{N}$ and centred on Euler angles $\rho_1, \rho_2 \in \mathrm{SO}(3)$, respectively, parameterising the rotation group $\mathrm{SO}(3)$, is denoted $\Xi^{(jj')}(\rho_1,\rho_2)$. We show that for any $j, j' \in \mathbb{N}$ such that $|j-j'| < 2$ and for any $\xi \in \mathbb{R}_*^+$, $\xi \geq 2N$ (where N is the azimuthal band-limit of the wavelet), there exists $C_1^{(j)}, C_2^{(j)} \in \mathbb{R}_*^+$ such that the directional wavelet correlation satisfies the bound:

$$\Xi^{(jj')}(\rho_1,\rho_2) \leq \frac{C_1^{(j)}}{\big(1+C_2^{(j)}\beta\big)^{\xi}} \,, \tag{2}$$