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Signal analysis based on complex wavelet signs

Martin Storath [a,*], Laurent Demaret [b,c], Peter Massopust [c]

[a] Biomedical Imaging Group, École Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland
[b] Institute of Computational Biology, Helmholtz Zentrum München, Ingolstädter Landstrasse 1, 85764 Neuherberg, Germany
[c] Department of Mathematics, Technische Universität München, Boltzmannstrasse 3, 85747 Garching bei München, Germany

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ABSTRACT

We propose a signal analysis tool based on the sign (or the phase) of complex wavelet coefficients, which we call a signature. The signature is defined as the fine-scale limit of the signs of a signal's complex wavelet coefficients. We show that the signature equals zero at sufficiently regular points of a signal whereas at salient features, such as jumps or cusps, it is non-zero. At such feature points, the orientation of the signature in the complex plane can be interpreted as an indicator of local symmetry and antisymmetry. We establish that the signature rotates in the complex plane under fractional Hilbert transforms. We show that certain random signals, such as white Gaussian noise and Brownian motions, have a vanishing signature. We derive an appropriate discretization and show the applicability to signal analysis.

1. Introduction

The determination and discrimination of salient features, such as jumps or cusps, is a frequently occurring task in signal and image processing [20,27,29,34,36]. In many classical approaches, it is assumed that the interesting features of a signal are located at the points of low regularity. In this context, local regularity is measured in terms of the (fractional) differentiability order, e.g., in the sense of local Hölder, Sobolev or Besov regularity [1,6,19,34]. Since such measures of smoothness only rely on the modulus of wavelet

* Corresponding author.
E-mail addresses: martin.storath@epfl.ch (M. Storath), laurent.demaret@mail.de (L. Demaret), massopust@ma.tum.de (P. Massopust).

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coefficients, they do not take into account sign (or phase) information. However, classical results in Fourier and wavelet analysis suggest that the signs of the wavelet coefficients contain rich information about a signal's structure. Logan [30] has shown that bandpass signals are characterized, up to a constant, by their zero crossings, which in turn are determined by the sign changes. The well known results by Oppenheim and Lim [37] indicate that sign information is even more important for the reconstruction of images than the modulus. If the amplitude of the Fourier transform of an image is combined with the sign of the Fourier transform of another image then the resulting reconstruction is structurally much closer to the second image, whereas the structure of the first one is hardly visible. Similar observations have been made for complex wavelet coefficients [15,40]. It has been shown by Jaffard [21,22] that a bounded signal becomes unbounded almost surely if the signs of its wavelet coefficients are randomly perturbed. This suggests that perturbations of the wavelet coefficients' signs can significantly alter the shape of a signal. But since the perturbation only affects the sign information this change is not taken into account by a purely modulus-based signal analysis. Let us illustrate this by an example of Meyer [35]. Consider the functions $f(x) = \operatorname{sgn} x$ and $g(x) = 2\log|x|$. Since f and g are related by the Hilbert transform, their wavelet coefficients are equal with respect to their order of magnitude. Although the singularities of f and g are structurally very different, they are not delineated by a purely modulus-based signal analysis.

Despite their high information content, the signs of the wavelet coefficients have received less attention than the moduli in signal and image analysis. First indications for the usability of wavelet signs in signal analysis have been given by Kronland-Martinet, Morlet, and Grossmann [28]. They observed that the lines of constant sign in the wavelet domain converge towards the singularities. Reconstruction algorithms from sign/phase information have been presented in [33] for wavelet coefficients and [45] for the short time Fourier transform. The phase of the short time Fourier transform has been recently analyzed in [2,23]. In [36,46], phase congruency was introduced for signal analysis based on the sign information of Fourier coefficients. Kovesi [27] added the multiscale aspect to phase congruency using log-Gabor wavelets. Phase congruency is especially useful for contrast invariant edge detection [27]. It has also been shown that phase congruency provides valuable information about local symmetries of a feature point, e.g. symmetric cusps or antisymmetric steps [27,36,46]. Similar observations have been made by Holschneider for the fine scale behavior of the wavelet phase [17, Ch. 4.3]. On the flipside, phase congruency lacks so far a rigorous foundation.

In this paper, we propose a new approach to signal analysis based on the sign (or phase) of complex wavelet coefficients. Our analyzing wavelets belong to the class of complex wavelets; this means that their real part and their imaginary part are Hilbert transforms of each other. Such analyzing functions can be traced back to the seminal work of Gabor [11]. They have been applied successfully to many signal and image processing applications [10,25,26,39,42]. Here, we investigate the fine scale behavior of the signs of complex wavelet coefficients. More precisely, we consider the complex-valued quantity

$$\sigma f(b) = \lim_{a\to 0^+} \operatorname{sgn}\langle f, \kappa_{a,b}\rangle = \lim_{a\to 0^+} \frac{\langle f, \kappa_{a,b}\rangle}{|\langle f, \kappa_{a,b}\rangle|},$$

where f is a real-valued signal, κ is a complex wavelet, $a > 0$ denotes the scale, and $b \in \mathbb{R}$ the location. If the limit does not exist, we set $\sigma f(b)$ to 0. We call the quantity $\sigma f(b)$ the *signature of f at location b*. The basic idea is that a nonzero signature indicates the presence of a salient point of the signal f, e.g. a step or a cusp, whereas at regular points, the signature equals zero. We first show that the signature is equal to zero if the signal f is locally polynomial around b. We then establish that the signature is ± 1 for cusp singularities of power type whereas it is equal to $\pm i$ for step singularities. Thus, the orientation of the signature within the complex plane may be interpreted as an indicator of local symmetry or antisymmetry. We further show that the singular support, which consists of all points where the signal is not locally C^∞, does in general not coincide with the support of signature. Thus, a singularity in the classical sense need not coincide with a singularity in the sense of signature. We further show that Gaussian white noise and fractional Brownian motion contain no salient points in the sense of the signature. We propose a suitable discretization and

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