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On small deformations of balanced manifolds ☆

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ABSTRACT

We introduce a property of compact complex manifolds under which the existence of balanced metric is stable by small deformations of the complex structure. This property, which is weaker than the $\partial\overline{\partial}$-Lemma, is characterized in terms of the strongly Gauduchon cone and of the first $\partial\overline{\partial}$-degree measuring the difference of Aeppli and Bott–Chern cohomologies with respect to the Betti number b_1.

0. Introduction

In this note we are aimed at the problem of constructing special metrics on complex non-Kähler manifolds. In particular, we are interested in *balanced metrics* in the sense of M.L. Michelsohn [17], that is, Hermitian metrics whose fundamental form is co-closed. More precisely, following the work by J. Fu and S.-T. Yau [13], we introduce a condition ensuring the existence of such metrics on small deformations of the complex structure.

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It is well known that the existence of balanced metrics on a compact complex manifold is not stable under small deformations of the complex structure [2]. More precisely, [2, Proposition 4.1] provides a counter-example on the Iwasawa manifold endowed with the holomorphically parallelizable complex structure. In fact, in order to prove a balanced analogue of the fundamental stability result by K. Kodaira and D.C. Spencer [14, Theorem 15], one needs a further assumption on the variation of Bott–Chern cohomology. We then investigate cohomological conditions on the central fibre yielding existence of balanced metrics for small deformations.

As a first result in this direction, C.-C. Wu proves in [25, Theorem 5.13] that small deformations of compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma and admitting balanced metrics still admit balanced metrics.

The assumption on the validity of $\partial\overline{\partial}$-Lemma is in fact stronger than necessary. It suffices, for example, that the dimension of the $(n-1,n-1)$-th Bott–Chern cohomology group is constant along the deformation, where $2n$ denotes the real dimension of the manifold, see Proposition 4.1. This condition, too, is sufficient but not necessary, see Example 4.10. The above condition allows us to show that any small deformation of the Iwasawa manifold endowed with an Abelian complex structure admits balanced metrics, see Proposition 4.4, a result that is in deep contrast with the behaviour of its holomorphically parallelizable structure.

In order to generalize Wu's result, J. Fu and S.-T. Yau introduced in [13, Definition 5] the following finer notion. A compact complex manifold X of complex dimension n is said to satisfy the $(n-1,n)$-*th weak* $\partial\overline{\partial}$-*Lemma* if, for each real form α of type $(n-1,n-1)$ on X such that $\overline{\partial}\alpha$ is ∂-exact, then there exists a $(n-2,n-1)$-form β such that $\overline{\partial}\alpha = \mathrm{i}\,\partial\overline{\partial}\beta$. They proved the following result.

Theorem 0.1 *([13, Theorem 6]). Let X be a compact complex manifold of complex dimension n with a balanced metric, and let X_t be a holomorphic deformation of $X = X_0$. If X_t satisfies the $(n-1,n)$-th weak $\partial\overline{\partial}$-Lemma for any $t \neq 0$, then there exists a balanced metric on X_t for t sufficiently close to 0.*

Notice that, while satisfying the $\partial\overline{\partial}$-Lemma is a stable property under small deformations of the complex structure, (see [24, Proposition 9.21], or [25, Theorem 5.12], or [7, Corollary 2.7]), satisfying the $(n-1,n)$-th weak $\partial\overline{\partial}$-Lemma is not open under deformations, (see [23, Example 3.7]). Here we propose a cohomological notion, related to the above weak $\partial\overline{\partial}$-Lemma, in order to get stability under small deformations.

Let X be a compact complex manifold of complex dimension n. We recall that the Bott–Chern and the Aeppli cohomologies [1,9] of X are defined, respectively, by

$$H^{\bullet,\bullet}_{BC}(X) := \frac{\ker\partial \cap \ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}} \qquad \text{and} \qquad H^{\bullet,\bullet}_{A}(X) := \frac{\ker\partial\overline{\partial}}{\operatorname{im}\partial + \operatorname{im}\overline{\partial}} .$$

Consider the natural map

$$\iota^{n-1,\,n}_{BC,\,A} \colon H^{n-1,\,n}_{BC}(X) \to H^{n-1,\,n}_{A}(X)$$

induced by the identity.

We introduce the following notion.

Definition 2.1. A compact complex manifold X of complex dimension n is said to satisfy the $(n-1,n)$-*th strong* $\partial\overline{\partial}$-*Lemma* if the natural map $\iota^{n-1,\,n}_{BC,\,A}$ is injective.

It is clear that compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma also satisfy the $(n-1,n)$-th strong $\partial\overline{\partial}$-Lemma. In Proposition 2.2, we prove that the latter property is weaker than the $\partial\overline{\partial}$-Lemma. It is also clear that the $(n-1,n)$-th strong $\partial\overline{\partial}$-Lemma implies the $(n-1,n)$-th weak $\partial\overline{\partial}$-Lemma. Proposition 2.2 shows that the converse does not hold.