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Dirac tori

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ABSTRACT

We consider conformal immersions $f : T^2 \to \mathbb{R}^3$ with the property that $H^2 f^* g_{\mathbb{R}^3}$ is a flat metric. These so called Dirac tori have the property that their Willmore energy is uniformly distributed over the surface and can be obtained using spin transformations of the plane by eigenvectors of the standard Dirac operator for a fixed eigenvalue. We classify Dirac tori and determine the conformal classes realized by them. Note that the spinors of Dirac tori satisfy the same system of PDE's as the differential of Hamiltonian stationary Lagrangian tori in $\mathbb{R}^4$. These were classified in [4].

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1. Introduction

Dirac surfaces are defined to be conformal immersions from a (compact) Riemann surface M into $\mathbb{R}^3$ such that the metric $H^2 f^* g_{\mathbb{R}^3}$ has constant Gaußian curvature, where H is the mean curvature and $f^* g_{\mathbb{R}^3}$ is the first fundamental form of the surface. For compact and oriented surfaces the sign of the Gaußian curvature is already determined by the topology of the surface and there are three cases to consider. The universal covering of M is either a sphere, a plane or the hyperbolic plane. In the case of positive Gaußian curvature we thus obtain Dirac spheres which are well understood by now, see [1]. The negative Gaußian curvature case is more difficult to handle since the hyperbolic plane cannot be isometrically immersed into $\mathbb{R}^3$ by a theorem of Hilbert. In this note we want to consider the case of vanishing Gaußian curvature, where the compact Riemann surface M is a torus and the universal covering of M is the plane. We show that Dirac surfaces in general can be obtained by spin transformations of a reference surface – an immersed surface of constant Gaußian and mean curvature in $\mathbb{R}^3$ inducing the background metric g_0 – by eigenvectors of the corresponding Dirac operator – using a variant of the Weierstrass representation for conformal immersions. For tori the reference surface is the plane and the metric $H^2 f^* g_{\mathbb{R}^3}$ is isometric to a constant multiple of the

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standard metric $g_{\mathbb{R}^2}$. Therefore the density $H|df|$ is constant for Dirac tori. This leads to the nice geometric property of Dirac tori that its Willmore energy given by $\int_M H^2|df|^2$ is uniformly distributed over the surface with respect to the flat metric $g_{\mathbb{R}^2}$ of the torus $T^2 \cong \mathbb{R}^2/\Gamma$. The quantity $H|df|$ is called the mean curvature half density.

In this note we first explain the general setup of spin transformations and how eigenvectors of Dirac operator conformally transform a reference surface, see [2] for details on quaternionic theory. Then we show that the Dirac surface property can be translated into an eigenvalue problem and derive the corresponding differential equations. Thereafter, we compute the double periodic spin transformations and compute the closing conditions to obtain closed Dirac tori. Moreover, we show that conformal classes actually provide Dirac tori satisfying a rationality condition.

The author would like to thank Ulrich Pinkall for helpful discussions. Dirac tori was also considered in [6].

2. Classification

Definition. A conformal immersion $f : M \to \mathbb{R}^3$ from a Riemann surface M is called a Dirac surface, if $H^2 f^* g_{\mathbb{R}^3}$ has constant Gaußian curvature, where H is the mean curvature and $f^* g_{\mathbb{R}^3}$ is the first fundamental form of the surface f.

Remark 1. The property of being a Dirac surface is scale invariant. Further, if M is a compact Riemann surface, the topology of the surface determines the sign of the Gaußian curvature of the metric $H^2 f^* g_{\mathbb{R}^3}$.

We consider the euclidean 3-space as the space of purely imaginary quaternions $\operatorname{Im}\mathbb{H} := \{a \in \mathbb{H} | a+\bar{a} = 0\}$. In this picture the quaternionic multiplication coincides with the cross product of $\mathbb{R}^3$ and any stretch rotation R of $\mathbb{R}^3$ is given by

$$R(x) = \bar{\lambda} x \lambda, \tag{2.1}$$

for an appropriate $\lambda \in \mathbb{H}$ and vice versa. The quaternions $\mathbb{H}$ can be turned into a complex vector space by choosing the right multiplication with $\mathbb{i}$ to be the complex structure. Thus we can identify $\mathbb{H} = \mathbb{C} \oplus \mathbb{j}\mathbb{C}$ and every $\lambda \in \mathbb{H}$ can be written as $\lambda = \lambda_1 + \mathbb{j}\lambda_2$ for two complex numbers λ_1 and λ_2.

Let $\lambda : M \to \mathbb{H}$ be a quaternionic valued function. Then we consider the 1-form

$$\eta = \bar{\lambda} df \lambda.$$

On a simply connected domain η integrates to a surface $\tilde{f}$ if and only if $d\eta = 0$. Following [5], we call this new map $\tilde{f}$ a *spin transformation* of f. A reformulation of the condition $d\eta = 0$ is stated in the lemma below, which can be found in [5,3].

Lemma 1 *([5,3]). Let $f : U \subset M \to \mathbb{R}^3$ be a conformal immersion and let $\eta = \bar{\lambda} df \lambda$. The 1-form η locally integrates to a conformal immersion $\tilde{f}$ if and only if there exists a real valued function ρ with*

$$D\lambda = \rho\lambda,$$

where

$$D\lambda := -\frac{df \wedge d\lambda(X,Y)}{vol_f(X,Y)}, \quad \textit{for tangential vector fields } X, Y$$

is the Dirac operator of the immersion f and vol_f is the induced volume form.

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