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A Myers-type compactness theorem by the use of Bakry–Emery Ricci tensor

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A B S T R A C T

Let (M, g) be a complete and connected Riemannian manifold of dimension $n \geq 2$. By using the Bakry–Emery Ricci curvature tensor on M, we prove a Myers-type compactness theorem which corresponds to the compactness theorem proved by Cheeger–Gromov–Taylor.

1. Introduction

In [2], J. Cheeger, M. Gromov and M. Taylor proved that, on a complete and connected n-dimensional Riemannian manifold (M, g), if the original Ricci curvature tensor $\mathrm{Ric}(x)$ has the lower bound

$$\mathrm{Ric} \geq (n-1)\frac{(\frac{1}{4}+\nu^2)}{r^2} \tag{1}$$

for all $x \in M$ satisfying the inequality $r(x) \geq r_0$ where $r_0 > 0$ and $\nu > 0$ are positive constants and the distance function r is defined with respect to a fixed point $p \in M$, i.e., $r(x) = d(x, p)$, then M is compact. Moreover, the diameter from the point $p \in M$ has an upper bound depending on the constants π, $\exp(1) = e$, ν and r_0 (see Theorem 4.8. in [2]). This compactness theorem proved in [2] is an extension of the Myers's classical theorem [6].

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In this direction, L.F. Wang [8] considered the m-Bakry–Emery Ricci tensor defined by

$$\mathrm{Ric}_{f,m} = \mathrm{Ric} + \mathrm{Hess} f - \frac{\mathrm{d}f \otimes \mathrm{d}f}{m-n} \tag{2}$$

where $m \geq n$ and the case $m = n$ implies that f is constant. He assumed

$$\mathrm{Ric}_{f,m}(x) \geq -(m-1)\frac{K_0}{(1+r(x))^2}, \quad \text{where } K_0 < -\frac{1}{4} \tag{3}$$

holding for any $x \in M$ (see [8], p. 188). Under this assumption, he proved a compactness theorem including a diameter bound (see Theorem 1.1 in [8]).

In the following theorem, we use the Bakry–Emery Ricci tensor [1] (also see [5,9]) and prove that M is compact such that diameter from a fixed $p \in M$ has a similar upper bound to obtained in [2]. For some compactness theorems proved by using the Bakry–Emery Ricci tensor, one can see [3,4,9,10].

The main result of this paper is the following theorem:

Theorem 1.1. *Let (M, g) be a complete and connected Riemannian manifold of dimension n, and let r be the distance function $r(x) = d(x, p)$ with respect to a fixed point $p \in M$. Assume that there is a smooth function $\psi \in \mathcal{C}^\infty(M)$ satisfying the inequalities*

$$\mathrm{Ric} + Hess(\psi) \geq (n-1)\frac{(\frac{1}{4}+\nu^2)}{r^2} \tag{4}$$

and

$$|\ \psi\ | \leq (n-1)c \tag{5}$$

for all $x \in M$ such that $r(x) \geq r_0$, where the constants r_0, c and ν satisfy the inequalities $r_0 > 0$, $c \geq 0$ and $\nu > 0$. Then M is compact and the diameter from the point $p \in M$ satisfies

$$\mathrm{diam}_p(M) \leq r_0 \exp\left(\frac{2}{\nu^2}\left(2c^2 + \frac{\pi^2\nu^2}{4} + 2c\sqrt{c^2 + \pi^2\nu^2\left(\frac{1}{4}+\nu^2\right)}\ \right)^{\frac{1}{2}}\right). \tag{6}$$

When $c = 0$, we obtain $\psi = 0$ and so Bakry–Emery Ricci tensor returns to the original Ricci tensor, i.e., $\mathrm{Ric}_\psi = \mathrm{Ric}$. Thus diameter estimate (6) has the same form as the J. Cheeger, M. Gromov and M. Taylor's estimate.

In order to prove Theorem 1.1, we utilize the index form I of a minimizing unit speed geodesic segment (see [7]).

2. Proof of the theorem

Let us first recall the definitions of gradient and Hessian of any smooth function $f \in \mathcal{C}^\infty(M)$ on a Riemannian manifold. The gradient and Hessian are defined by $g(\nabla f, V) = V(f)$ and $\big(\mathrm{Hess}(f)\big)(V, W) = g(\nabla_V \nabla f, W)$ for all vector field V, W, respectively. For a distance function $r(x) = d(x, p)$ where $p \in M$ is a fixed point, it is well-known that r is only smooth on $M - \big(C_p \cup \{p\}\big)$ where C_p denotes the cut-locus of the point $p \in M$. In addition to this fact, we have $\nabla r = \partial_r$ in the adapted coordinates with respect to the r, and also have $g(\nabla r, \nabla r) = 1$ where r is smooth.