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Remarks on the space of volume preserving embeddings

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ABSTRACT

Let (N, g) be a Riemannian manifold. Given a compact, connected and oriented submanifold M of N, we define the space of volume preserving embeddings $\mathrm{Emb}_\mu(M, N)$ as the set of smooth embeddings $f : M \hookrightarrow N$ such that $f^*\mu^f = \mu$, where μ^f (resp. μ) is the Riemannian volume form on $f(M)$ (resp. M) induced by the ambient metric g (the orientation on $f(M)$ being induced by f).
In this article, we use the Nash–Moser inverse function Theorem to show that the set of volume preserving embeddings in $\mathrm{Emb}_\mu(M, N)$ whose mean curvature is nowhere vanishing forms a tame Fréchet manifold, and determine explicitly the Euler–Lagrange equations of a natural class of Lagrangians.
As an application, we generalize the Euler equations of an incompressible fluid to the case of an "incompressible membrane" of arbitrary dimension moving in N.

0. Introduction

Fluid mechanics and infinite dimensional geometry already share a long and common history. In 1966, Arnold [3] suggested to regard the space of velocity fields of an incompressible fluid as the Lie algebra of the infinite dimensional Lie group of volume preserving diffeomorphisms:

$$\mathrm{SDiff}_\mu(M) := \left\{ \phi \in \mathrm{Diff}(M) \,|\, \phi^*\mu = \mu \right\}. \tag{1}$$

Here M is the oriented manifold on which the fluid is living, μ is the volume form of M and $\mathrm{Diff}(M)$ is the group of all smooth diffeomorphisms of M. In this setting, Arnold interpreted the Euler equations of an incompressible fluid as a geodesic equation on $\mathrm{SDiff}_\mu(M)$ for an appropriate right-invariant metric.

It was not until the 70's that Arnold's vision of fluid mechanics could be made partially rigorous with the development of Banach and Hilbert manifolds. In [6], Ebin and Marsden considered volume preserving diffeomorphisms on a compact manifold M which are not smooth, but of Sobolev classes. In doing so, they obtained topological groups locally modelled on Hilbert spaces, and were able, following Arnold's ideas, to

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prove analytical results on the Euler equations. Their method is still an active research area (see for example [7,8,17]).

On the geometrical side, volume preserving diffeomorphisms which are not smooth are problematic. For, the left-multiplication $L_\phi : \mathrm{Diff}(M) \to \mathrm{Diff}(M)\,,\ \psi \mapsto \phi \circ \psi$ consumes derivatives of ϕ, and thus, subgroups of the group of diffeomorphisms whose elements are *not* smooth cannot be turned into genuine infinite dimensional Lie groups (left multiplication is not smooth). Hence, from a Lie group theoretical point of view, one has to consider the group of *smooth* volume preserving diffeomorphisms of (M, μ), i.e., the group $\mathrm{SDiff}_\mu(M)$.

For technical reasons, $\mathrm{SDiff}_\mu(M)$ can only be given a Lie group structure modelled on topological vector spaces which are more general than Banach and Hilbert spaces, and an inverse function theorem, applicable beyond the usual Banach space category, is necessary. To our knowledge, only two authors succeeded in doing this. The first was Omori who showed and used an inverse function theorem in terms of ILB-spaces ("inverse limit of Banach spaces", see [19]), and later on, Hamilton with his category of tame Fréchet spaces together with the Nash–Moser inverse function Theorem (see [11]). Nowadays, it is nevertheless not uncommon to find mistakes or big gaps in the literature when it comes to the differentiable structure of $\mathrm{SDiff}_\mu(M)$, even in some specialized textbooks in infinite dimensional geometry. The case of M being non-compact is even worse, and no proof that $\mathrm{SDiff}_\mu(M)$ is a "Lie group" is available in this case.

A natural generalization of $\mathrm{SDiff}_\mu(M)$, with which we shall be concerned in this paper, is the space of *volume preserving embeddings* $\mathrm{Emb}_\mu(M, N)$. This space is defined as follows. For a Riemannian manifold (N, g) and a compact, connected and oriented submanifold M of N,

$$\mathrm{Emb}_\mu(M, N) := \left\{ f \in \mathrm{Emb}(M, N) \,\middle|\, f^*\mu^f = \mu \right\}, \tag{2}$$

where $\mathrm{Emb}(M, N)$ is the space of smooth embeddings from M into N, and where μ^f (resp. μ) is the Riemannian volume form on $f(M)$ (resp. M) induced by the ambient metric g (the orientation on $f(M)$ being induced by f).

When M is an open subset of $\mathbb{R}^n$ with boundary,[1] then it is possible to extend Arnold's method by introducing a L^2-metric on $\mathrm{Emb}_\mu(M, N)$ and to show that the corresponding geodesics describe the dynamics of a liquid drop with free boundary. This has been discussed formally in [16], and rigorous results in this direction can be obtained using spaces of volume preserving embeddings of Sobolev classes, as pointed out to us by Sergiy Vasylkevych.[2]

In this paper, we focus on *smooth* volume preserving embeddings, i.e., on the space $\mathrm{Emb}_\mu(M, N)$ as defined above. To this end, we adopt a rigorous infinite dimensional point of view based on Hamilton's category of tame Fréchet manifolds, and determine explicitly a natural class of Lagrangian equations on $\mathrm{Emb}_\mu(M, N)$. We allow M to be of arbitrary dimension, and we assume that it has no boundary.

More precisely, using the techniques developed by Hamilton in [11], as well as a generalization of the Helmholtz–Hodge decomposition Theorem for vector fields supported on submanifolds (Proposition 1.4), we are able, in Theorem 1.6, to show the following result: *the space $Emb_\mu(M, N)^\times$ of volume preserving embeddings whose mean curvature is nowhere vanishing forms a tame Fréchet submanifold of $Emb(M, N)$.* This result is a consequence of the Nash–Moser inverse function Theorem.

In [9], a stronger statement using other technics is made, namely that $\mathrm{Emb}_\mu(M, N)$ itself is a Fréchet manifold (no condition is imposed on the mean curvature). There is, however, a gap, and an attentive reading of the proof of [9, Proposition 3.3], which is crucial, shows that the mean curvature cannot be identically zero. In the very recent paper [5], it is also shown among other results that the set of volume preserving

[1] In this paper, all manifolds have no boundary.
[2] Private communication.