ARTICLE IN PRESS

Journal of Complexity ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Complexity

journal homepage: www.elsevier.com/locate/jco

Super-polynomial convergence and tractability of multivariate integration for infinitely times differentiable functions

Kosuke Suzuki *

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

ARTICLE INFO

Article history:
Received 8 February 2016
Accepted 11 October 2016
Available online xxxx
Communicated by E. Novak

Keywords:
Numerical integration
Tractability
Super-polynomial convergence
Quasi-Monte Carlo
Walsh spaces
Digital nets

ABSTRACT

We investigate multivariate integration for a space of infinitely times differentiable functions $\mathcal{F}_{s,\boldsymbol{u}} := \{f \in C^\infty[0,1]^s \mid \|f\|_{\mathcal{F}_{s,\boldsymbol{u}}} < \infty\}$, where $\|f\|_{\mathcal{F}_{s,\boldsymbol{u}}} := \sup_{\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_s)\in\mathbb{N}_0^s} \|f^{(\boldsymbol{\alpha})}\|_{L^1} / \prod_{j=1}^s u_j^{\alpha_j}$, $f^{(\boldsymbol{\alpha})} := \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1}\cdots\partial x_s^{\alpha_s}} f$ and $\boldsymbol{u} = \{u_j\}_{j\ge 1}$ is a sequence of positive decreasing weights. Let $e(n,s)$ be the minimal worst-case error of all algorithms that use n function values in the s-variate case. We prove that for any $\boldsymbol{u}$ and s considered $e(n,s) \le C(s)\exp(-c(s)(\log n)^2)$ holds for all n, where $C(s)$ and $c(s)$ are constants which may depend on s. Further we show that if the weights $\boldsymbol{u}$ decay sufficiently fast then there exist some $1 < p < 2$ and absolute constants C and c such that $e(n,s) \le C\exp(-c(\log n)^p)$ holds for all s and n. These bounds are attained by quasi-Monte Carlo integration using digital nets. These convergence and tractability results come from those for the Walsh space into which $\mathcal{F}_{s,\boldsymbol{u}}$ is embedded.

1. Introduction

In this paper we approximate the integral on an s-dimensional unit cube

$$\int_{[0,1)^s} f(\boldsymbol{x})\, d\boldsymbol{x}$$

* Correspondence to: School of Mathematics and Statistics, The University of New South Wales, Sydney, NSW 2052, Australia.
E-mail address: kosuke.suzuki1@unsw.edu.au.

http://dx.doi.org/10.1016/j.jco.2016.10.002
0885-064X/

Please cite this article in press as: K. Suzuki, Super-polynomial convergence and tractability of multivariate integration for infinitely times differentiable functions, Journal of Complexity (2016), http://dx.doi.org/10.1016/j.jco.2016.10.002

by a quasi-Monte Carlo (QMC) algorithm which uses n function values of the form

$$A_{n,s}(f) := \sum_{i=1}^{n} \frac{1}{n} f(\boldsymbol{t}_i) \quad \text{for } \boldsymbol{t}_i \in [0,1)^s.$$

One classical issue is the optimal rate of convergence with respect to n. Another important issue is the dependence on the number of variables s, since s can be hundreds or more in computational applications. The latter issue is related to the notion of tractability if we require no exponential dependence on s.

A large number of studies have been devoted to numerical integration on the unit cube for various function spaces. One typical case is that functions are only finitely many times differentiable, e.g., functions with bounded variation, periodic functions in the Korobov space and non-periodic functions in the Sobolev space, see [14,18,16,5] and the references therein. For these cases, it is known that the rate of convergence is $O(n^{-\alpha})$ for some $\alpha > 0$ and thus we have polynomial convergence. Another interesting case is when the functions are smooth, i.e., infinitely times differentiable. Dick [2] gave reproducing kernel Hilbert spaces based on Taylor series for which higher order QMC rules achieve a convergence of $O(n^{-\alpha})$ with $\alpha > 0$ arbitrarily large. The spaces were later generalized in [23]. Further results were proved in [4,10], where it is shown that exponential convergence holds for the Korobov space of periodic functions whose Fourier coefficients decay exponentially fast. Exponential convergence means that the integration error converges as $O(q^{n^p})$ for some $q \in (0,1)$, $p > 0$. Note that exponential convergence was also shown for Hermite spaces on $\mathbb{R}^s$ with exponentially fast decaying Hermite coefficients [9].

In this paper we focus on a weighted normed space of non-periodic smooth functions

$$\mathcal{F}_{s,\boldsymbol{u}} := \left\{ f \in C^\infty[0,1]^s \mid \|f\|_{\mathcal{F}_{s,\boldsymbol{u}}} := \sup_{\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_s)\in\mathbb{N}_0^s} \frac{\|f^{(\boldsymbol{\alpha})}\|_{L^1}}{\prod_{j=1}^{s} u_j^{\alpha_j}} < \infty \right\} \tag{1}$$

with a sequence of positive weights $\boldsymbol{u} = \{u_j\}_{j\geq 1}$, where $f^{(\boldsymbol{\alpha})} := \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1}\cdots\partial x_s^{\alpha_s}} f$. It is easy to check that all functions in $\mathcal{F}_{s,\boldsymbol{u}}$ are analytic from Taylor's theorem. For $s = 1$, it is known that the integration error using Gaussian quadrature with n points converges factorially, see, for instance, [1, (6.53)]. Our interest is the multivariate QMC rules using so-called digital nets. This is motivated by the results by Yoshiki [22] and is closely related to the notion of Walsh figure of merit (WAFOM) [12,20,22] first introduced by Matsumoto, Saito and Matoba. WAFOM is a criterion for numerical integration using digital nets and is computable in a reasonable time. Hence we can search for good digital nets with respect to WAFOM by computer, see [12,7,6] for numerical experiments. We observe that generalized WAFOM works well for the space $\mathcal{F}_{s,\boldsymbol{u}}$, see Remark 6.4.

The first purpose of this paper is to show that the integration error of QMC rules using digital nets for $\mathcal{F}_{s,\boldsymbol{u}}$ can achieve super-polynomial convergence as $O(\exp(-c(\log n)^2)) \asymp O(n^{-c\log n})$ for all s and $\boldsymbol{u}$ considered. Here the hidden constant and c may depend on s. We remark that this convergence behavior was first observed in [13] as the decay of the lowest-WAFOM value and that the combination of [13,22] implies the convergence result for $\mathcal{F}_{s,(1/2)_{j\geq 1}}$.

We also consider tractability for $\mathcal{F}_{s,\boldsymbol{u}}$. Let us briefly recall the notion of tractability (see [15–17] for more information). Let $n(\varepsilon, s)$ be the information complexity, i.e., the minimal number n of function values which are required in order to approximate the s-variate integration within ε. An integration problem is said to be tractable if $n(\varepsilon, s)$ does not grow exponentially in ε nor s. In particular, two notions of tractability have been mainly considered: polynomial tractability, i.e., $n(\varepsilon, s) \leq C\varepsilon^{-\tau_1}s^{\tau_2}$, and strong polynomial tractability, i.e., $n(\varepsilon, s) \leq C\varepsilon^{-\tau_1}$ for $\tau_1, \tau_2 \geq 0$. A common way to obtain tractability is to consider weighted function spaces as introduced by Sloan and Woźniakowski [19]. Weighted spaces here mean that the dependence on the successive variables can be moderated by weights. Our weights $\boldsymbol{u}$ play the same role. For tractability results for spaces of smooth functions, see also [8].

Please cite this article in press as: K. Suzuki, Super-polynomial convergence and tractability of multivariate integration for infinitely times differentiable functions, Journal of Complexity (2016), http://dx.doi.org/10.1016/j.jco.2016.10.002