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Robin problems with a general potential and a superlinear reaction

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Abstract

We consider semilinear Robin problems driven by the negative Laplacian plus an indefinite potential and with a superlinear reaction term which need not satisfy the Ambrosetti–Rabinowitz condition. We prove existence and multiplicity theorems (producing also an infinity of smooth solutions) using variational tools, truncation and perturbation techniques and Morse theory (critical groups).

MSC: Primary 35J20, 35J60; Secondary 58E05

Keywords: Indefinite potential; Robin boundary condition; Superlinear reaction term; Regularity theory; Critical groups; Nodal solutions

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1. Introduction

In this paper we study the following semilinear elliptic problem with Robin boundary condition:

$$\left\{\begin{array}{l} -\Delta u(z)+\xi(z)u(z)=f(z,u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n}+\beta(z)u=0 \text{ on } \partial\Omega. \end{array}\right\} \tag{1}$$

In this problem $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$. The potential function $\xi \in L^s(\Omega)$ with $s > N$ is in general sign-changing. So, the linear part of (1) is indefinite. The reaction term $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z,x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto f(z,x)$ is continuous), which exhibits superlinear growth near $\pm\infty$. However, $f(z,\cdot)$ does not satisfy the (usual in such cases) Ambrosetti–Rabinowitz condition (AR-condition, for short). Instead, we employ a more general condition which incorporates in our framework superlinear functions with "slower" growth near $\pm\infty$, which fail to satisfy the AR-condition. Another nonstandard feature of our work is that $f(z,\cdot)$ does not have subcritical polynomial growth. In our case, the growth of $f(z.\cdot)$ is almost critical in the sense that $\lim\limits_{x\to\pm\infty}\dfrac{f(z,x)}{|x|^{2^*-2}x}=0$ uniformly for almost all $z \in \Omega$, with 2^* being the Sobolev critical exponent for 2, defined by

$$2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \geq 3 \\ +\infty & \text{if } N = 1,2. \end{cases}$$

In the boundary condition, $\dfrac{\partial u}{\partial n}$ denotes the normal derivative of $u \in H^1(\Omega)$ defined by extension of the continuous linear map

$$C^1(\overline{\Omega}) \ni u \mapsto \frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient is $\beta \in W^{1,\infty}(\partial\Omega)$ and we assume that $\beta(z) \geq 0$ for all $z \in \partial\Omega$. When $\beta \equiv 0$, we have the usual Neumann problem.

Our aim in this paper is to prove existence and multiplicity results within this general analytical framework. Recently, there have been such results primarily for Dirichlet problems. We mention the works of Lan and Tang [14] (with $\xi \equiv 0$), Li and Wang [15], Miyagaki and Souto [17] (with $\xi \equiv 0$), Papageorgiou and Papalini [21], Qin, Tang and Zhang [29], Wu and An [34], Zhang–Liu [35]. For Neumann and Robin problems, we mention the works of D'Agui, Marano and Papageorgiou [5], Papageorgiou and Rădulescu [23,24,26], Papageorgiou, Rădulescu and Repovš [27], Papageorgiou and Smyrlis [28], Pucci et al. [2,4], Shi and Li [31]. Superlinear problems were treated by Lan and Tang [14], Li and Wang [15], Miyagaki and Souto [17], who proved only existence results. The superlinear case was not studied in the context of Neumann and Robin problems.

Our approach uses variational methods based on the critical point theory, together with suitable truncation and perturbation techniques and Morse theory (critical groups).

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