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Nonexistence of nonconstant steady-state solutions in a triangular cross-diffusion model

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Abstract

In this paper we study the Shigesada–Kawasaki–Teramoto model for two competing species with triangular cross-diffusion. We determine explicit parameter ranges within which the model exclusively possesses constant steady state solutions.

MSC: 35B36; 35J57

Keywords: Cross-diffusion; Competition; Steady state

1. Introduction

We consider the problem

$$\begin{cases} \Delta[(d_1 + a_{11}u + a_{12}v)u] + u(1 - u - a_1 v) = 0, & x \in \Omega, \\ d_2 \Delta v + v(1 - v - a_2 u) = 0, & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

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in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with smooth boundary, where $d_1 > 0$, $d_2 > 0$, $a_{12} \geq 0$, $a_{11} \geq 0$, $a_1 \geq 0$ and $a_2 \geq 0$ are parameters, and where $\nu = \nu_\Omega$ denotes the outward normal vector field on $\partial\Omega$. Our goal consists in identifying parameter ranges within which the system (1.1) exclusively possesses spatially homogeneous solutions. Thinking of u and v as representing population densities, we will focus here on nonnegative solutions, where throughout the sequel by a *nonnegative classical solution* of (1.1) we mean a pair $(u, v) \in (C^1(\overline{\Omega}) \cap C^2(\Omega))^2$ which is such that $u \geq 0$ and $v \geq 0$ in $\overline{\Omega}$, and that each identity in (1.1) is satisfied in the pointwise sense.

System (1.1) is a special case of the Shigesada–Kawasaki–Teramoto model (abbreviated as SKT henceforth), which was proposed in [13] to describe the spatial segregation of two competing species. There have been extensive studies on the existence of non-constant positive steady states of the SKT model; See [2–9,12,14–16]. Surveys of the SKT model can be found in [10, 11,17]. Some of these works also investigate the non-existence of non-constant positive steady states of the SKT model. For instance, in the weak competition case (i.e., $a_1 < 1$ and $a_2 < 1$), it is shown in [4] that when $a_{11} = 0$, system (1.1) has no non-constant positive solution if one of the following three quantities is small: a_{12}/d_1, a_{12}/d_2, $a_{12}/\sqrt{d_1 d_2}$. The main goal of the paper is to provide *explicit* bounds for various parameters such that system (1.1) has no non-constant positive solution. These results will facilitate further understanding of the global dynamics of system (1.1).

Main results. The first class of our main results in this direction addresses the case when the quantity u is not influenced by any self-diffusion in the sense that the coefficient a_{11} in the first equation from (1.1) vanishes. In this situation, we shall establish two sufficient criteria for nonexistence of inhomogeneous solutions in cases when one of the competition parameters exceeds the critical value 1, whereas the other does not. According to the asymmetry in (1.1) induced by the assumption therein that only the first population is capable of cross-diffusive migration, our respective assumptions (1.3) and (1.6) on the further system parameters will be substantially different: When the second population has a competitive advantage in that (1.2) holds, we will only require the smallness condition (1.3) on its diffusion rate, without any restriction on the strength of cross-diffusion; in the opposite case (1.5), however, our hypothesis (1.6) will additionally involve the cross-diffusion parameter a_{12}. More precisely:

Theorem 1.1. *Suppose that $a_{11} = 0$, $a_{12} \geq 0$, $d_1 > 0$, $d_2 > 0$, $a_1 \geq 0$ and $a_2 \geq 0$, and that (u, v) is a nonnegative classical solution of (1.1).*

i) If

$$a_1 > 1 > a_2, \tag{1.2}$$

and if moreover

$$d_1 \geq d_2, \tag{1.3}$$

then

$$\text{either } (u, v) \equiv (0, 0), \quad \text{or } (u, v) \equiv (1, 0), \quad \text{or } (u, v) \equiv (0, 1). \tag{1.4}$$

ii) In the case when

$$a_1 < 1 < a_2 \tag{1.5}$$

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