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Sharp estimates for the anisotropic torsional rigidity and the principal frequency

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ABSTRACT

In this paper we generalize some classical estimates involving the torsional rigidity and the principal frequency of a convex domain to a class of functionals related to some anisotropic nonlinear operators.

1. Introduction

Let h_K be the norm associated to a convex body K (see Section 3 for more details); given a domain $\Omega \subset \mathbb{R}^N$ with finite measure, we define the K-principal frequency, λ_1^K, and the K-torsional rigidity, T^K, as

$$\lambda_1^K(\Omega) = \min_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega h_K^2(\nabla u)\,dx}{\int_\Omega u^2\,dx}, \tag{1.1}$$

and

$$T^K(\Omega) = \max_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\left(\int_\Omega u\,dx\right)^2}{\int_\Omega h_K^2(\nabla u)\,dx}. \tag{1.2}$$

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It is convenient to introduce the function $H_K = h_K^2/2$; when H_K is sufficiently smooth, we can write the *Euler–Lagrange equations* for the minimizers of the problems (1.1) and (1.2) to get a PDE interpretation of the above quantities. Indeed, the K-principal frequency is related to the eigenvalue problem

$$-\Delta_K u = \lambda_1^K u \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{1.3}$$

while the K-torsional rigidity is the L^1 norm of the solution u of:

$$-\Delta_K u = 1 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{1.4}$$

Here Δ_K denotes the *Finsler–Laplace operator* given by

$$\Delta_K u = \operatorname{div}(DH_K(\nabla u)). \tag{1.5}$$

In the Euclidean case, occurring when K is the unitary ball B, (and $h_K(x) = |x|$) the operator given in (1.5) coincides with the Laplacian and λ_1 and T are the usual first Dirichlet eigenvalue and torsional rigidity.

As in the linear case, the quantities defined in (1.1) and (1.2) are monotone, in opposite sense, with respect to the set inclusion, *i.e.* if $\Omega_1 \subset \Omega_2$ then

$$\lambda_1^K(\Omega_1) \geq \lambda_1^K(\Omega_2) \quad \text{and} \quad T^K(\Omega_1) \leq T^K(\Omega_2). \tag{1.6}$$

Moreover, since H_K is a homogeneous function of degree 2, the following scalings hold true:

$$\lambda_1^K(t\Omega) = t^{-2}\lambda_1^K(\Omega) \qquad \text{and} \qquad T^K(t\Omega) = t^{N+2}T^K(\Omega), \quad t > 0. \tag{1.7}$$

Shape optimization problems involving λ_1 and T, or even more general spectral functionals of the form $\mathcal{F}(\Omega) = \Phi(\lambda_1(\Omega), T(\Omega))$, are widely studied in the literature (see for instance [2–6,11,12]) and, as it is well known, it is possible to get both lower and upper bounds for the principal frequency and the torsional rigidity in terms of quantities associated to the geometry of the domain Ω, such as, for instance the perimeter and the volume (just think to the *Faber–Krahn inequality* and the *Saint-Venant theorem*, see for instance the recent book [11]).

As it should not be unexpected, if we impose some further constraints in the class of admissible domains, we can get stronger estimates. The class of convex domains, for instance, has been considered by several authors: on one hand the *a priori* assumption of the convexity of the domain naturally arises in many situations; on the other, the class of convex sets has strong compactness properties which ensure the existence of extremal domains for a great number of geometric inequalities.

In this paper we are interested in estimates of the principal frequency and the torsional rigidity of a convex domain in terms of the *inradius*, R_Ω, *i.e.* the radius of the biggest ball contained in Ω.

An immediate consequence of (1.6) and (1.7) is that, for the Euclidean case

$$\lambda_1(\Omega) \leq \lambda_1(B_{R_\Omega}) = \lambda_1(B_1)R_\Omega^{-2}. \tag{1.8}$$

A classical result by J. Hersch (see [13]) shows that for any convex domain $\Omega \subset \mathbb{R}^2$ it holds

$$\frac{\pi^2}{4}R_\Omega^{-2} \leq \lambda_1(\Omega), \tag{1.9}$$

and the inequality is sharp: if we allow unbounded domains, equality case occurs when Ω is a strip, otherwise it is reached only asymptotically, by a sequence of rectangles with sides $a \ll b$. Hersch's technique has been extended to convex domains of $\mathbb{R}^N$ by M.H. Protter in [16] who proved the validity of (1.9) in every dimension.

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