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Optimal monopoly pricing with congestion and random utility via partial mass transport

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A R T I C L E I N F O

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A B S T R A C T

We consider a bilevel optimization framework corresponding to a monopoly spatial pricing problem: the price for a set of given facilities maximizes the profit (upper level problem) taking into account that the demand is determined by consumers' cost minimization (lower level problem). In our model, both transportation costs and congestion costs are considered, and the lower level problem is solved via partial transport mass theory. The partial transport aspect of the problem comes from the fact that each consumer has the possibility to remain out of the market. We also generalize the model and our variational analysis to the stochastic case where utility involves a random term.

1. Introduction

Since the classical work of Hotelling [5], spatial pricing issues have received a lot of attention. Many generalizations and variants of Hotelling's competitive model where firms compete both in locations and prices have been studied in literature (see e.g. [9] and the references therein). In the present paper, we consider a monopoly situation but allow for general transport costs, congestion effects and possible randomness in the consumers' utility.

In our model, there is a fixed finite set of locations at which the monopoly can sell a homogeneous good to a continuum of consumers, distributed according to a given spatial distribution μ. Our aim is to analyze profit maximizing spatial pricing. The profit maximization can naturally (as in Mallozzi and Passarelli di Napoli [8]) be viewed as a special instance of bilevel optimization. Indeed, consumer's demands at each facility location is determined by their cost minimizing behavior, based not only on price but also on traveling cost and congestion or queuing (as in Crippa, Jimenez and Pratelli [2]) effects. We call the

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consumer's demand stage, given the price system, the lower level and the profit maximization with respect to the price the upper level. The lower level problem can be seen as an equilibrium partition problem, in the same spirit as the generalized market area problem [7,10] where the production levels and the distribution patterns at n plants are determined simultaneously to satisfy the demand distributed over a given region.

Our analysis of the lower level problem with congestion is very similar to the variational/mass transport approach of [2], with one important difference in the fact that, in our model, we do not impose that the market is fully covered, i.e. that the total demand is the mass of μ. Indeed, in our model, consumers have a reservation cost, corresponding to the option of not purchasing the good anywhere and then paying zero cost. It may then well be the case that some consumers remain out of the market and this effect is actually even strengthened by congestion effects. It is also important to allow the market not to be fully covered since it might be too costly for the monopoly hence non-optimal for the upper level problem. We show nevertheless that the analysis of [2] easily extends to the not covered case provided one allows partial optimal transport (see for instance Figalli [4] for a detailed analysis of partial optimal transport, in particular for a quadratic cost). The importance of partial optimal transport in optimal/equilibrium partition problems was clearly emphasized in the recent work of Wolansky [11] who introduced a new cooperative approach to partition games (but did not consider congestion effects). This quite general framework enables us to go one step further and prove an existence result for the upper level. Note that, in our upper level problem, the demands for some facilities can vanish, so if we imagine that the finite set of feasible facilities for the monopoly is a very fine discretization of the whole urban region, the upper level problem also determines the effective optimal operating locations for the monopoly. Deeper theoretical or numerical investigations of optimal prices are left for future research.

Most realistic economic situations involve some stochastic effects (see e.g. [6], for a random utility scheme in a competitive facility problem). Another contribution of our paper is to allow for some randomness (or heterogeneity) in consumers' utilities and to show how the variational approach to the lower level problem can be extended to this noisy setting.

The organization of the paper is the following: the model is described in section 2 and some tractable examples are presented in section 3. The lower level problem is shown to be equivalent to a convex variational problem in section 4, we deduce an existence result for the monopolit's upper level problem in section 5. Our analysis is extended to the random utility case in section 6. Some technical results from optimal partial transport and convex duality are gathered in Appendix A.

2. The model

We consider an urban area given by $\Omega \subset \mathbb{R}^d$, a bounded domain (i.e. open connected) of $\mathbb{R}^d$, the density of population/customers in this region is given by a probability measure $\mu \in \mathcal{P}(\overline{\Omega})$ which captures the potential spatial distribution of demand. We are interested in the profit-maximizing pricing policies of a monopoly operating at N given distinct locations $y_1, \cdots, y_N \in \overline{\Omega}^N$. Each customer is assumed to purchase either 1 or 0 quantity of the good sold by the monopoly at one of the locations $y_1, \cdots, y_N$. The demand for the good at each location $y_1, \cdots, y_N$ results from the cost-minimizing behavior of customers which we now describe. First (and this is in contrast with the model of [2] for instance), we assume that customers also have the option of not purchasing the good then getting a reservation cost of 0. If, on the contrary, a customer from x decides to purchase the good from the monopoly at y_j, her cost will be the sum of a transport cost $c(x, y_j)$, a congestion (or queuing cost) cost $h_j(\omega_j)$ where ω_j is the demand at location j net of a utility u_j for purchasing the good at a price p_j. Prices p_j and demands ω_j's are the main unknowns to be determined from the monopoly and customers' rational behaviors.

In addition to the city Ω and the locations $y_1, \cdots, y_N$, the data of the model are the transport cost c, the customers distributions μ, the congestion functions h_j, the vector of utilities $\boldsymbol{u} := (u_1, \cdots, u_N)$ (in the