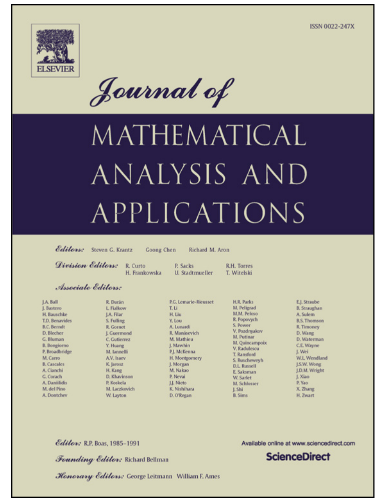

On the comparison principle for unbounded solutions of elliptic equations with first order terms

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ON THE COMPARISON PRINCIPLE FOR UNBOUNDED SOLUTIONS OF ELLIPTIC EQUATIONS WITH FIRST ORDER TERMS

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Abstract. We prove a comparison principle for unbounded weak sub/super solutions of the equation

$$\lambda u - \operatorname{div}(A(x)Du) = H(x, Du) \quad \text{in } \Omega$$

where $A(x)$ is a bounded coercive matrix with measurable ingredients, $\lambda \geq 0$ and $\xi \mapsto H(x,\xi)$ has a super linear growth and is convex at infinity. We improve earlier results where the convexity of $H(x,\cdot)$ was required to hold globally.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain and let $A(x) = (a_{i,j}(x))$ be a coercive matrix of $L^\infty(\Omega)$ functions. This note is concerned with the uniqueness of *unbounded* solutions to the elliptic problem

$$\begin{cases} \lambda u - \operatorname{div}(A(x)Du) = H(x, Du) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $H(x, Du)$ is measurable with respect to x, locally Lipschitz with respect to ξ and has a super linear growth with respect to the gradient, namely

$$|H(x,\xi)| \leq \gamma|\xi|^q + f(x) \qquad \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^N, \tag{1.2}$$

for some $q > 1$ and $f(x)$ belonging to some Lebesgue space $L^m(\Omega)$, which will be detailed later.

It is well known that, if $\xi \mapsto H(x,\xi)$ is locally Lipschitz and has at most linear growth, then problem (1.1) admits a unique weak solution in the Sobolev space $H^1_0(\Omega)$. This is no longer true in case of super linear growth of the first order terms, and uniqueness may fail. For example, the function

$$u(x) = c_{q,N}\left(|x|^{-\frac{2-q}{q-1}} - 1\right) \tag{1.3}$$

is a nontrivial solution of the problem

$$\begin{cases} -\Delta u = |Du|^q & \text{in } B_1(0), \\ u = 0 & \text{on } \partial B_1(0), \end{cases} \tag{1.4}$$

in the distributional sense, if $N/(N-1) < q < 2$ and for a suitable choice of the constant $c_{q,N} > 0$. In particular, this is also a non trivial $H^1_0(\Omega)$ solution if $1 + \frac{2}{N} < q < 2$.

This shows that the comparison principle does not hold straightforwardly for elliptic equations with super linear first order terms in the class of unbounded solutions, so this issue should be handled with care.

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