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Imputing a variational inequality function or a convex objective function: A robust approach

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ABSTRACT

To impute the function of a variational inequality and the objective of a convex optimization problem from observations of (nearly) optimal decisions, previous approaches constructed inverse programming methods based on solving a convex optimization problem [17,7]. However, we show that, in addition to requiring complete observations, these approaches are not robust to measurement errors, while in many applications, the outputs of decision processes are noisy and only partially observable from, *e.g.*, limitations in the sensing infrastructure. To deal with noisy and missing data, we formulate our inverse problem as the minimization of a weighted sum of two objectives: 1) a duality gap or Karush–Kuhn–Tucker (KKT) residual, and 2) a distance from the observations robust to measurement errors. In addition, we show that our method encompasses previous ones by generating a sequence of Pareto optimal points (with respect to the two objectives) converging to an optimal solution of previous formulations. To compare duality gaps and KKT residuals, we also derive new sub-optimality results defined by KKT residuals. Finally, an implementation framework is proposed with applications to delay function inference on the road network of Los Angeles, and consumer utility estimation in oligopolies.

1. Introduction

1.1. Motivation

Many decision processes are modeled as a Variational Inequality (VI) or Convex Optimization (CO) problem [15,9]. However, the function that describes these processes are often difficult to estimate while their outputs (the decisions they describe) are often directly observable. For example, the traffic assignment

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problem considers a road network in which each road segment is associated to a delay that is a function of the volume of traffic on the arc [22]. The Wardrop's equilibrium principles [25] describe an equilibrium flow that is easily locally measurable by induction loop detectors or video cameras. While the delay functions are in general not observable, having accurate estimates of these functions is still crucial for urban planning. However, due to their cost of maintenance, traffic sensors are sparse, we thus present an approach robust to missing values and measurement errors. In consumer utility estimation, for example, the consumer is assumed to purchase various products from different companies in order to maximize a utility function minus the price paid, where the utility function measures the satisfaction the consumer receives from his purchases. In practice, the consumer's utility function is difficult to estimate but the consumer purchases, which is a function of the products' prices, are easily observable. We refer to [17,7] for more examples, *e.g.*, value function estimation control.

1.2. Contributions and outline

Estimating the parameters of a process based on observations is related to various lines of work, *e.g.*, inverse reinforcement learning in robotics [20,1], the inverse shortest path problem [10], recovering the parameters of the Lyapunov function given a linear control policy [8, §10.6]. The field of *structural estimation* in economics estimates the parameters of observed equilibrium models, *e.g.* imputing production and demand functions [23,2,4]. In general, *inverse problems* have been studied quite extensively and we refer to [17,7] for more references on the subject. In [17] (resp. [7]), a program is proposed to impute a convex objective (resp. a VI function) based on complete observations of nearly optimal decisions. The program is solved via CO.

After reviewing preliminary results in VI and CO in Section 2 and formally stating the problem in Section 3, our contributions in the remainder of the present article is as follows. In Section 4, we demonstrate that the methods presented in [17,7] are in general not robust to noise and outliers in the data. In Section 5, we formulate our inverse problem as a weighted sum of a distance r_{obs} from the observations and residual functions r_{eq} in the form of duality gaps or Karush–Kuhn–Tucker (KKT) residuals, and show that our method is robust to noise and outliers while it avoids the disjunctive nature of the complementary condition. In Section 6, we show that the proposed weighted sum defines a set of Pareto efficient points whose closure contains a solution to the programs proposed in [17,7]. Our method thus encompasses previous ones but performs better against noise and missing data. It also provides a conceptual way to recognize the implicit assumption of full noiseless observations made by previous inverse programming approaches. In Section 7, we compare the KKT residual and the duality gap and derive new sub-optimality results defined by the KKT residuals. In Section 8, an implementation framework is proposed. Finally, we apply our method to delay inference in the road network of Los Angeles, and consumer utility estimation and pricing in oligopolies in Sections 9 and 10.

2. Preliminaries

2.1. Variational inequality (VI) and convex optimization (CO)

VI is used to model a broad class of problems from economics, convex optimization, and game theory, see, *e.g.* [15], for a comprehensive treatment of the subject. Mathematically, a VI problem is defined as follows:

Definition 2.1. Given a closed, convex set $\mathcal{K} \subseteq \mathbb{R}^n$ and a map $F : \mathcal{K} \to \mathbb{R}^n$, the VI problem, denoted $\text{VI}(\mathcal{K}, F)$, consists in finding a vector $\mathbf{x} \in \mathcal{K}$ such that

$$F(\mathbf{x})^T(\mathbf{u} - \mathbf{x}) \geq 0, \forall\, \mathbf{u} \in \mathcal{K} \tag{1}$$

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