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Convergence rates of Neumann problems for Stokes systems

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ABSTRACT

In quantitative homogenization of the Neumann problems for Stokes systems with rapidly oscillating periodic coefficients, this paper studies the convergence rates of the velocity in L^2 and H^1 as well as those of the pressure term in L^2, without any smoothness assumptions on the coefficients.

1. Introduction and main results

In this paper, we would like to investigate the convergence rates of Neumann problems for Stokes systems with rapidly oscillating periodic coefficients. Specifically, we'd like to consider the following Neumann problem for Stokes system in a bounded domain $\Omega \subset \mathbb{R}^d$,

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = F & \text{in } \Omega, \\ \operatorname{div}(u_\varepsilon) = g & \text{in } \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} - p_\varepsilon n = f & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where n denotes the outward unit normal to $\partial\Omega$. Throughout this paper, we use the summation convention and let $\varepsilon > 0$ be a small parameter. We define the second-order elliptic operator in divergence form $\mathcal{L}_\varepsilon$ associated with coefficient matrix A by

$$\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon)\nabla) = -\frac{\partial}{\partial x_i}\left[a_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right] \tag{1.2}$$

where $1 \le i, j, \alpha, \beta \le d$, and the conormal derivative of system (1.1) on $\partial\Omega$ is defined by

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$$\left(\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right)^\alpha - p_\varepsilon\, n_\alpha = n_i(x) a_{ij}^{\alpha\beta}(x/\varepsilon)\frac{\partial u_\varepsilon^\beta}{\partial x_j} - p_\varepsilon(x) n_\alpha(x). \tag{1.3}$$

We assume that the coefficient matrix $A(y) = (a_{ij}^{\alpha\beta}(y))$ is real, bounded measurable, and it satisfies the ellipticity condition:

$$\mu|\xi|^2 \le a_{ij}^{\alpha\beta}(y)\xi_i^\alpha \xi_j^\beta \le \frac{1}{\mu}|\xi|^2 \qquad \text{for } y \in \mathbb{R}^d \text{ and } \xi = (\xi_i^\alpha) \in \mathbb{R}^{d\times d}, \tag{1.4}$$

where $\mu > 0$, and also the periodicity condition,

$$A(y+z) = A(y) \qquad \text{for } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d. \tag{1.5}$$

A function satisfying (1.5) will be called 1-periodic.

The homogenization theory of Neumann problems for Stokes systems tells us that, $u_\varepsilon - \fint_\Omega u_\varepsilon$ converges to $u_0 - \fint_\Omega u_0$ weakly in H^1, and $p_\varepsilon - \fint_\Omega p_\varepsilon$ converges to $p_0 - \fint_\Omega p_0$ weakly in L^2, given suitable F, f and g. Here $(u_0, p_0) \in H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega)$ is the weak solution of the associated homogenized problem with constant coefficients,

$$\begin{cases} \mathcal{L}_0(u_0) + \nabla p_0 = F & \text{in } \Omega, \\ \operatorname{div}(u_0) = g & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial \nu_0} - p_0\, n = f & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

The nature and primary question will be how fast does it converge. Our main purpose is to study the optimal convergence rate of $\|u_\varepsilon - u_0\|_{L^2(\Omega)}$, as $\varepsilon \to 0$. The result is given in the following theorem.

Theorem 1.1. *Let Ω be a bounded $C^{1,1}$ domain. Suppose A satisfies ellipticity condition (1.4) and periodicity condition (1.5). Given $F \in L^2(\Omega;\mathbb{R}^d)$ and $f \in H^{1/2}(\partial\Omega;\mathbb{R}^d)$ satisfying the compatibility condition*

$$\int_\Omega F + \int_{\partial\Omega} f = 0, \tag{1.7}$$

for any $g \in H^1(\Omega)$, let $(u_\varepsilon, p_\varepsilon)$, (u_0, p_0) be the weak solutions of Neumann problems (1.1), (1.6), respectively. If $\int_\Omega u_\varepsilon = \int_\Omega u_0 = 0$, then

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{1.8}$$

where the constant C depends only on μ, d, and Ω.

Theorem 1.1 gives us the order $O(\varepsilon)$ convergence of the velocity in L^2, which is optimal in the sense of $\|u_0\|_{H^2(\Omega)}$. The other important result of this paper, which is shown in the next theorem, is that the two-scale expansion of $(u_\varepsilon, p_\varepsilon)$ has optimal $O(\varepsilon^{1/2})$ rates in $H^1 \times L^2$.

For simplicity, we will use the notation $h^\varepsilon(x) = h(x/\varepsilon)$, for any function h. Here (χ, π) are the correctors associated with A, defined as in (2.5), and S_ε is the Steklov smoothing operator introduced in (2.12).

Theorem 1.2. *Let Ω be a bounded $C^{1,1}$ domain. Suppose A satisfies ellipticity condition (1.4) and periodicity condition (1.5). Let $(u_\varepsilon, p_\varepsilon)$ and (u_0, p_0) be the same as in Theorem 1.1. If $\int_\Omega u_\varepsilon = \int_\Omega u_0 = 0$, then*

$$\|u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon S_\varepsilon\left(\nabla\widetilde{u}_0\right)\|_{H^1(\Omega)} \le C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}, \tag{1.9}$$