J. Math. Anal. Appl. 457 (2018) 498–521

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A uniqueness criterion of limit cycles for planar polynomial systems with homogeneous nonlinearities ☆

Jianfeng Huang [a], Haihua Liang [b,*]

[a] Department of Mathematics, Jinan University, Guangzhou 510632, PR China
[b] School of Mathematics and Systems Science, Guangdong Polytechnic Normal University, Guangzhou 510665, PR China

ARTICLE INFO

Article history:
Received 20 December 2016
Available online 24 August 2017
Submitted by Y. Huang

Keywords:
Limit cycles
Uniqueness
Polynomial differential systems
Homogeneous nonlinearities

ABSTRACT

This paper is devoted to study the planar polynomial system:

$$\dot{x} = ax - y + P_n(x, y), \quad \dot{y} = x + ay + Q_n(x, y),$$

where $a \in \mathbb{R}$ and P_n, Q_n are homogeneous polynomials of degree $n \geq 2$. Denote $\psi(\theta) = \cos(\theta) \cdot Q_n\big(\cos(\theta), \sin(\theta)\big) - \sin(\theta) \cdot P_n\big(\cos(\theta), \sin(\theta)\big)$. We prove that the system has at most 1 limit cycle surrounding the origin provided $(n-1)a\psi(\theta) + \dot{\psi}(\theta) \neq 0$. Furthermore, this upper bound is sharp. This is maybe the first uniqueness criterion, which only depends on a (linear) condition of ψ, for the limit cycles of this kind of systems. We show by examples that in many cases, the criterion is applicable while the classical ones are invalid. The tool that we mainly use is a new estimate for the number of limit cycles of Abel equation with coefficients of indefinite signs. Employing this tool, we also obtain another geometric criterion which allows the system to possess at most 2 limit cycles surrounding the origin.

1. Introduction and statements of main results

One of the significant problems in the qualitative theory of real planar differential systems is to control the number of limit cycles for a given class of polynomial systems, which is originated from the second part of Hilbert's 16th problem.

In this paper we restrict our study to the number of limit cycles surrounding the origin for the planar polynomial system with homogeneous nonlinearities:

☆ The first author is supported by the NSFC (No. 11401255) and the China Scholarship Council (No. 201606785007), the Fundamental Research Funds for the Central Universities (No. 21614325) and the China Scholarship Council (No. 201606785007). The second author is supported by the NSFC (No. 11771101) and Natural Science Foundation of Guangdong Province (No. 2015A030313669).

* Corresponding author.
E-mail addresses: thuangjf@jnu.edu.cn (J. Huang), lianghhgdin@126.com (H. Liang).

http://dx.doi.org/10.1016/j.jmaa.2017.08.008

$$\begin{cases} \dfrac{dx}{dt} = ax - y + P_n(x,y), \\ \dfrac{dy}{dt} = x + ay + Q_n(x,y), \end{cases} \tag{1}$$

where P_n, Q_n are homogeneous polynomials of degree $n \geq 2$.

As we know, (1) is a system which has been extensively studied and gained wide attention in decades. One of the particularities of this system is that each limit cycle surrounding the origin can be expressed in polar coordinates as $r = r(\theta)$, with $r(\theta)$ being a smooth periodic function, see for instance [11], [14], [18] and [19], etc. This particularity provides us an opportunity to consider the Hilbert's 16th problem in a natural and simple way.

So far, plenty of works have been carried out for the bifurcation of (1) with small perturbations, see for instance [5], [15], [19], [24], [25], [29], [33] and the references therein. In contrast, only a few results for the non-bifurcation case are obtained. Here we summarize the representative ones as below: Let

$$\begin{aligned} \varphi(\theta) &= \cos(\theta) \cdot P_n\big(\cos(\theta), \sin(\theta)\big) + \sin(\theta) \cdot Q_n\big(\cos(\theta), \sin(\theta)\big), \\ \psi(\theta) &= \cos(\theta) \cdot Q_n\big(\cos(\theta), \sin(\theta)\big) - \sin(\theta) \cdot P_n\big(\cos(\theta), \sin(\theta)\big). \end{aligned} \tag{2}$$

(I) If $\varphi(\theta) - a\psi(\theta) \not\equiv 0$ does not change sign, then (1) has at most 1 limit cycle surrounding the origin (see Coll, Gasull and Prohens [14]).
(II) If $\psi(\theta)\big(\varphi(\theta) - a\psi(\theta)\big) \not\equiv 0$ does not change sign, then (1) has at most 1 (resp. 2) limit cycle(s) surrounding the origin when n is even (resp. odd) (see Carbonell and Llibre [11]).
(III) If $(n-1)\big(\varphi(\theta) - 2a\psi(\theta)\big) - \dot{\psi}(\theta) \not\equiv 0$ does not change sign, then (1) has at most 2 limit cycles surrounding the origin (see Gasull and Llibre [18]).
(IV) If either $(n-1)\big(\varphi(\theta) - 2a\psi(\theta)\big) - \dot{\psi}(\theta) \equiv 0$, or $\psi(\theta)\big(\varphi(\theta) - a\psi(\theta)\big) \equiv 0$, then (1) has at most 1 limit cycle surrounding the origin (see Gasull and Llibre [18]).

There are several powerful tools to study system (1) in the papers mentioned above. One of them is the Abel equation

$$\dot{x} = \frac{dx}{dt} = S(t,x) = a_3(t)x^3 + a_2(t)x^2 + a_1(t)x, \tag{3}$$

where $x \in \mathbb{R}$ and $a_i \in C^\infty([0,1])$, $i = 1,2,3$.

In fact, system (1) in polar coordinates can be written in the form

$$\frac{d}{dt}(\theta, r)^T = v \triangleq \left(1 + r^{n-1}\psi(\theta), ar + r^n\varphi(\theta)\right)^T. \tag{4}$$

It is known that the limit cycles surrounding the origin of system (1) do not intersect the curve $1 + r^{n-1}\psi(\theta) = 0$ (see [11], [14], [18], etc). Therefore, these limit cycles can be investigated by equation

$$\frac{dr}{d\theta} = \frac{ar + r^n\varphi(\theta)}{1 + r^{n-1}\psi(\theta)}, \quad \theta \in [0, 2\pi]. \tag{5}$$

Furthermore, using the transformation introduced by Cherkas [12]

$$\rho = \frac{r^{n-1}}{1 + r^{n-1}\psi(\theta)}, \quad \theta = 2\pi\tau, \tag{6}$$

equation (5) becomes an Abel equation