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Ultradistributional boundary values of harmonic functions on the sphere ☆

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A B S T R A C T

We present a theory of ultradistributional boundary values for harmonic functions defined on the Euclidean unit ball. We also give a characterization of ultradifferentiable functions and ultradistributions on the sphere in terms of their spherical harmonic expansions. To this end, we obtain explicit estimates for partial derivatives of spherical harmonics, which are of independent interest and refine earlier estimates by Calderón and Zygmund. We apply our results to characterize the support of ultradistributions on the sphere via Abel summability of their spherical harmonic expansions.

1. Introduction

The study of boundary values of harmonic and analytic functions is a classical and important subject in distribution and ultradistribution theory. There is a vast literature dealing with boundary values on $\mathbb{R}^n$, see e.g. [1,5,6,10,12,14,20] and references therein. In the case of the unit sphere $\mathbb{S}^{n-1}$, the characterization of harmonic functions in the Euclidean unit ball of $\mathbb{R}^n$ having distributional boundary values on $\mathbb{S}^{n-1}$ was given by Estrada and Kanwal in [11]. In a recent article [13], González Vieli has used the Poisson transform to obtain a very useful description of the support of a Schwartz distribution on the sphere (cf. [27] for support characterizations on $\mathbb{R}^n$). Representations of analytic functionals on the sphere [17] as initial values of solutions to the heat equation were studied by Morimoto and Suwa [18].

In this article we generalize the results from [11] to the framework of ultradistributions [15,16] and supply a theory of ultradistributional boundary values of harmonic functions on $\mathbb{S}^{n-1}$. Our goal is to characterize all those harmonic functions U, defined in the unit ball, that admit boundary values $\lim_{r\to 1^-} U(r\omega)$ in an ultradistribution space $\mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$. Our considerations apply to both non-quasianalytic and quasianalytic

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ultradistributions, and, in particular, to analytic functionals. As an application, we also obtain a characterization of the support of a non-quasianalytic ultradistribution in terms of Abel summability of its spherical harmonic series expansion. Since Schwartz distributions are naturally embedded into the spaces of ultradistributions in a support preserving fashion, our support characterization contains as a particular instance that of González Vieli quoted above.

In Section 4 we study spaces of ultradifferentiable functions and ultradistributions through spherical harmonics. Our main results there are descriptions of these spaces in terms of the decay or growth rate of the norms of the projections of a function or an ultradistribution onto the spaces of spherical harmonics. We also establish the convergence of the spherical harmonic series in the corresponding space. Note that eigenfunction expansions of ultradistributions on compact analytic manifolds have recently been investigated in [8,9] with the aid of pseudodifferential calculus (cf. [28] for the Euclidean global setting). However, our approach here is quite different and is rather based on explicit estimates for partial derivatives of solid harmonics and spherical harmonics that are obtained in Section 3. Such estimates are of independent interest and refine earlier bounds by Calderón and Zygmund from [4].

Harmonic functions with ultradistributional boundary values are characterized in Section 5. The characterization is in terms of the growth order of the harmonic function near the boundary $\mathbb{S}^{n-1}$; we also show in Section 5 that a harmonic function satisfying such growth conditions must necessarily be the Poisson transform of an ultradistribution. In the special case of analytic functionals, our result yields as a corollary: *any* harmonic function on the unit ball arises as the Poisson transform of some analytic functional on the sphere. Finally, Section 6 deals with the characterization of the support of non-quasianalytic ultradistributions on $\mathbb{S}^{n-1}$.

2. Preliminaries

We employ the notation $\mathbb{B}^n$ for the open unit ball of $\mathbb{R}^n$. We work in dimension $n \geq 2$.

2.1. Spherical harmonics

The theory of spherical harmonics is a classical subject in analysis and it is very well explained in several textbooks (see e.g. [2,3]). The space of solid harmonics of degree j will be denoted by $\mathcal{H}_j(\mathbb{R}^n)$, its elements are the harmonic homogeneous polynomials of degree j on $\mathbb{R}^n$. A spherical harmonic of degree j is the restriction to $\mathbb{S}^{n-1}$ of a solid harmonic of degree j and we write $\mathcal{H}_j(\mathbb{S}^{n-1})$ for space of all spherical harmonics of degree j. Its dimension, denoted as $d_j = \dim \mathcal{H}_j(\mathbb{S}^{n-1})$, is (cf. [3] or [26, Thm. 2, p. 117])

$$d_j = \frac{(2j+n-2)(n+j-3)!}{j!(n-2)!} \sim \frac{2j^{n-2}}{(n-2)!}.$$

From this exact formula, it is not hard to see that d_j satisfies the bounds

$$\frac{2}{(n-2)!} j^{n-2} < d_j \leq n j^{n-2}, \quad \text{for all } j \geq 1. \tag{2.1}$$

It is well known [3] that

$$L^2(\mathbb{S}^{n-1}) = \bigoplus_{j=0}^{\infty} \mathcal{H}_j(\mathbb{S}^{n-1}),$$

where the L^2-inner product is taken with respect to the surface measure of $\mathbb{S}^{n-1}$. The orthogonal projection of $f \in L^2(\mathbb{S}^{n-1})$ onto $\mathcal{H}_j(\mathbb{S}^{n-1})$ will always be denoted as f_j; it is explicitly given by