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Continuous approximation of linear impulsive systems and a new form of robust stability

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ABSTRACT

The time-scale tolerance for linear ordinary impulsive differential equations is introduced. How large the time-scale tolerance is directly reflects the degree to which the qualitative dynamics of the linear impulsive system can be affected by replacing the impulse effect with a continuous (as opposed to discontinuous, impulsive) perturbation, producing what is known as an impulse extension equation. Theoretical properties related to the existence of the time-scale tolerance are given for periodic systems, as are algorithms to compute them. Some methods are presented for general, aperiodic systems. Additionally, sufficient conditions for the convergence of solutions of impulse extension equations to the solutions of their associated impulsive differential equation are proven. Counterexamples are provided.

1. Introduction

Impulsive differential equations provide an elegant way to describe systems that undergo very fast changes in state [2,12,18]. These changes in state occur so quickly that they are idealized as discontinuities. Impulsive differential equations have a host of applications, including pulse vaccinations [1,8], seasonal skipping in recurrent epidemics [19], antiretroviral drug treatment [10,14] and birth pulses in animals [17].

Impulse extension equations have been put forward as a framework to study properties of impulsive differential equations that remain invariant if one replaces the impulse effect by a continuous perturbation [5]. Results on existence and uniqueness of solutions, as well as specialized results for linear periodic systems, have been developed [6,7].

In the present article, two similar but ultimately different problems are solved. First, given a linear impulsive differential equation, we associate to it a family of impulse extension equations that is parameterized by its step sequences. We then provide sufficient conditions under which the solutions of the impulse extension equation converge to those of the impulsive differential equation, as the step sequence becomes small. These

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sufficient conditions are then tied to results relating to stability of the family of impulse extension equations, relative to the impulsive differential equation that generate it.

Following this, the *time-scale tolerance* is introduced first for linear, periodic impulsive differential equations, and then in general linear systems. The time-scale tolerance behaves as a robust stability threshold; if the norm of a given step sequence is smaller than the time-scale tolerance, than all impulse extensions equations from a particular class will have the same stability classification as the associated impulsive differential equation. From the point of view of applications, this indicates that if an impulsive differential equation models some physical process, then the approximation by an impulsive differential equation is, in a certain sense, "valid", provided the perturbations that are idealized as impulses occur on a time-scale that is smaller than the time-scale tolerance. Methods to compute the time-scale tolerance are proposed.

2. Background material on impulse extension equations

Throughout this paper, we will be interested in continuous systems that approximate the linear, finite-dimensional impulsive differential equation,

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + g(t), && t \neq \tau_k, \\ \Delta x &= B_k x + h_k, && t = \tau_k, \end{aligned} \tag{1}$$

as well as its associated homogeneous equation,

$$\begin{aligned} \frac{dx}{dt} &= A(t)x, && t \neq \tau_k, \\ \Delta x &= B_k x, && t = \tau_k. \end{aligned} \tag{2}$$

It is assumed that the sequence of impulse times, $\{\tau_k\}$, is monotone increasing and unbounded. Also, we assume all functions appearing in the differential equations above are sufficiently regular to guarantee that for any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, there is a unique solution $x(t)$ defined on $[t_0, \infty)$ satisfying $x(t_0) = x_0$. For example, it suffices to have all functions be bounded and measurable on compact sets.

We now comment on some notation related to sequences that will be relevant. If $s = \{s_n\}$ is a real-valued sequence, we define $\Delta s_n = s_{n+1} - s_n$ to be the forward difference. Also, indexed families of sequences, such as $\{s^j : j \in U\}$ for some index set U, will always have their index appear in the exponent. As such, the symbol s^j_n indicates the nth element of the sequence s^j, for $j \in U$.

The following definition of an *impulse extension equation* for (2) is a modified version of that appearing in [7]; the present definition is for linear systems, and allows us to more concretely study the convergence of their solutions, which is necessary to fulfill the objective of this article.

Definition 2.1. Consider the linear impulsive differential equation (1).

- A *step sequence over* τ_k is sequence of positive real numbers $a = \{a_k\}$ such that $a_k < \Delta\tau_k$ for all $k \in \mathbb{Z}$. We denote $\mathcal{S}_j = \mathcal{S}_j(a) \equiv [\tau_j, \tau_j + a_j)$ and $\mathcal{S} = \mathcal{S}(a) \equiv \bigcup_{j\in\mathbb{Z}} \mathcal{S}_j$. The set of all step sequences will be denoted $\mathcal{S}^*$, and is defined by

$$\mathcal{S}^* \equiv \{a : \mathbb{Z} \to \mathbb{R}_+ \,,\, 0 < a_k < \Delta\tau_k\}.$$

- A sequence of functions $\varphi = \{(\varphi^B_k, \varphi^h_k)\}$,

$$\varphi^B_k : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^{n\times n}, \qquad \varphi^h_k : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^n,$$