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Fractal snowflake domain diffusion with boundary and interior drifts

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A R T I C L E I N F O

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A B S T R A C T

We study a parabolic Ventsell problem for a second order differential operator in divergence form and with interior and boundary drift terms on the snowflake domain. We prove that under standard conditions a related Cauchy problem possesses a unique classical solution and explain in which sense it solves a rigorous formulation of the initial Ventsell problem. As a second result we prove that functions that are intrinsically Lipschitz on the snowflake boundary admit Euclidean Lipschitz extensions to the closure of the entire domain. Our methods combine the fractal membrane analysis, the vector analysis for local Dirichlet forms and PDE on fractals, coercive closed forms, and the analysis of Lipschitz functions.

Contents

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1. Introduction

The main objective of this paper is to study a parabolic Ventsell problem for a second order differential operator in divergence form with measurable coefficients and drift terms in the interior and with drift and diffusion terms on the boundary of the classical snowflake domain Ω, see Fig. 1. This continues the investigation of diffusion problems related to fractal membranes started in [43] (see [9,47] for the most recent results and references).

A central tool in our analysis is the expression

$$\mathcal{E}_{\mathcal{A}}(f,f) = \int_{\Omega} (\mathcal{A}(x)\cdot\nabla f(x))\cdot\nabla f(x)\mathcal{L}^2(dx) + c_0\mathcal{E}_{\partial\Omega}(f,f),$$

which, together with a suitable domain of definition, is a local Dirichlet form. Here $\mathcal{A}$ is a bounded measurable uniformly elliptic two-by-two matrix valued function, $\mathcal{L}^2$ denotes the usual two dimensional Lebesgue measure, $c_0 > 0$ is a fixed constant, and $\mathcal{E}_{\partial\Omega}$ denotes the usual Kusuoka–Kigami Dirichlet form on the snowflake boundary $\partial\Omega$, which is a union of three copies of the classical Koch curve, [35,36,41,42]. Using perturbation arguments we can pass from $\mathcal{E}_{\mathcal{A}}$ to the bilinear form corresponding to the parabolic Ventsell problem that we wish to study. This problem can be formulated as

$$\begin{cases} u_t(t,x) - L_{\mathcal{A}}u(t,x) - \vec{b}(x)\cdot\nabla u(t,x) = f(t,x) & \text{in } (0,T]\times\Omega \\ u_t(t,x) - c_0\Delta_{\partial\Omega}u(t,x) - b_{\partial\Omega}(x)D_{\partial\Omega}u(t,x) + c(x)u(t,x) & \\ \qquad = -\dfrac{\partial u(t,x)}{\partial n_{\mathcal{A}}} + f(t,x) & \text{in } (0,T]\times\partial\Omega \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1}$$

Here $L_{\mathcal{A}}u(t,\cdot) = \operatorname{div}(\mathcal{A}\cdot\nabla u(t,\cdot))$, and $\Delta_{\partial\Omega}$ is the Kusuoka–Kigami Laplacian on $\partial\Omega$. The function f is a time-dependent external forcing, the function c is a stationary scalar potential on $\partial\Omega$, and u_0 is a given initial condition. The term $\frac{\partial u(t,\cdot)}{\partial n_{\mathcal{A}}}$ is the co-normal derivative of $u(t,\cdot)$ across $\partial\Omega$, to be defined in a suitable distributional sense, see for example [47] or Section 5. The symbol $\vec{b}$ denotes a stationary drift vector field in Ω, and $b_{\partial\Omega}$ is a drift vector field on the snowflake boundary $\partial\Omega$. This drift $b_{\partial\Omega}$ is defined using a special case of the intrinsic approach to derivatives for Dirichlet forms, see e.g. [10,26], applied to the form $\mathcal{E}_{\partial\Omega}$. We point out that $b_{\partial\Omega}$ does not have to be the trace of $\vec{b}$ on $\partial\Omega$. The expression $D_{\partial\Omega}u(t,\cdot)$ plays the role of the tangential derivative of $u(t,\cdot)$ along $\partial\Omega$. We prove that it can be defined as the measurable gradient on the fractal boundary $\partial\Omega$ of the restriction $u(t,\cdot)|_{\partial\Omega}$ of $u(t,\cdot)$ to $\partial\Omega$. In principle this

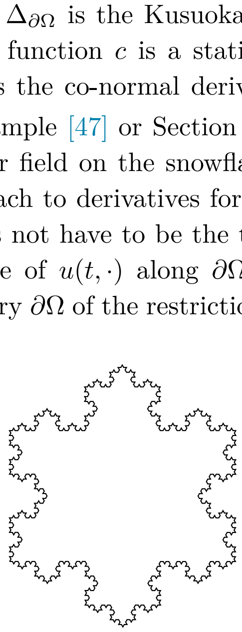

Fig. 1. Fractal (closed) snowflake domain Ω.