J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Bilinear forms on homogeneous Sobolev spaces ☆

Carme Cascante [a,b,*], Joan Fàbrega [a], Joaquín M. Ortega [a]

[a] Dept. Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via 585, 08071 Barcelona, Spain
[b] Barcelona Graduate School of Mathematics (BGSMath), Spain

ARTICLE INFO

Article history:
Received 20 February 2017
Available online xxxx
Submitted by P.G. Lemarie-Rieusset

Keywords:
Bilinear form
Sobolev spaces
Fractional laplacian
Riesz potential

ABSTRACT

In this paper we characterize the boundedness of the bilinear form defined by

$$(f,g) \in \dot{H}^s(\mathbb{R}) \times \dot{H}^s(\mathbb{R}) \to \int_{\mathbb{R}} (-\Delta)^{s/2}(fg)(x)(-\Delta)^{s/2}(b)(x)dx,$$

in the product of homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}) \times \dot{H}^s(\mathbb{R})$, $0 < s < 1/2$. We deduce a characterization of the space of pointwise multipliers from $\dot{H}^s(\mathbb{R})$ to its dual $\dot{H}^{-s}(\mathbb{R})$ in terms of trace measures.

1. Introduction

If $-d/2 < t$, and f is in the Schwartz class, $\mathcal{S}(\mathbb{R}^d)$, we define the fractional laplacian $(-\Delta)^t$ as the distribution defined by

$$(-\Delta)^t(f)(x) := \mathcal{F}^{-1}((2\pi|x|)^{2t}\mathcal{F}(f)(x)),$$

where $\mathcal{F}$ denotes the Fourier transform.

If $0 < t < d/2$, then $(-\Delta)^{-t}$ can be described in terms of the Riesz integral operator, that is, $(-\Delta)^{-t} = I_{2t}$, where

$$I_{2t}(f)(x) := c_{d,2t} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2t}} dy, \qquad f \in \mathcal{S}(\mathbb{R}^d),$$

for some positive constant $c_{d,2t}$.

☆ The research was supported in part by Ministerio de Economía y Competitividad, Spain, projects MTM2014-51834-P and MTM2015-69323-REDT, and Generalitat de Catalunya, project 2014SGR289. First author was supported in part by Ministerio de Economía y Competitividad, Spain, project MDM-2014-0445 (María de Maeztu).

* Corresponding author.
E-mail addresses: cascante@ub.edu (C. Cascante), joan_fabrega@ub.edu (J. Fàbrega), ortega@ub.edu (J.M. Ortega).

http://dx.doi.org/10.1016/j.jmaa.2017.08.033
0022-247X/

We recall that, for $0 < t < 1$, there is an equivalent definition of the fractional Laplacian in terms of singular integrals:

$$(-\Delta)^t f(x) := C_{d,t} P.V. \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x-y|^{d+2t}} dy, \qquad f \in \mathcal{S}(\mathbb{R}^d), \tag{1.1}$$

where $C_{d,t}$ is a positive constant and $P.V.$ means principal value.

The fractional homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$, $|s| < d/2$, is the completion of the space of compactly supported C^∞ functions on $\mathbb{R}^d$, $\mathcal{D}(\mathbb{R}^d)$, with respect to the norm

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} := \|(-\Delta)^{s/2}(f)\|_{L^2(\mathbb{R}^d)}.$$

One of the reasons to restrict ourselves to the case where $0 < 2s < d$ is Sobolev inequality that proves that in this case, $\dot{H}^s(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$, where $q = 2d/(d-2s)$. So $\dot{H}^s(\mathbb{R}^d)$ can be identified as the space of all the functions on $L^q(\mathbb{R}^d)$ such that $\|f\|_{\dot{H}^s(\mathbb{R}^d)} < \infty$.

It is well known that if $0 < 2s < d$, then $\dot{H}^s(\mathbb{R}^d) = I_s(L^2(\mathbb{R}^d))$.

These spaces satisfy $\mathcal{S}(\mathbb{R}^d) \subset \dot{H}^s(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ and, by Plancherel's formula, we also have that the dual of $\dot{H}^s(\mathbb{R}^d)$ is $\dot{H}^{-s}(\mathbb{R}^d)$ with the natural $L^2(\mathbb{R}^d)$-pairing.

Some classical references for the fractional laplacian $(-\Delta)^t$, are the books of [6,14] (see also the book of [8] and the references therein). The characterization of the fractional laplacian in terms of singular integrals can be found, for instance, in [5,15].

The main goal of this paper is the study of the boundedness of the bilinear form on $\dot{H}^s(\mathbb{R}^d)$ with symbol $b \in \dot{H}^s(\mathbb{R}^d)$ defined by

$$\Lambda_b(f,g) := \int_{\mathbb{R}^d} (-\Delta)^{s/2}(fg)(x)(-\Delta)^{s/2}(b)(x)dx, \quad f,g \in \mathcal{D}(\mathbb{R}^d).$$

This boundedness, via Plancherel's formula is equivalent to

$$\left| \int_{\mathbb{R}^d} (fg)(x)(-\Delta)^s(b)(x)dx \right| \lesssim \|f\|_{\dot{H}^s(\mathbb{R}^d)} \|g\|_{\dot{H}^s(\mathbb{R}^d)}.$$

Observe then that the boundedness of the bilinear form Λ_b is equivalent to the fact that $(-\Delta)^s(b)$ is a pointwise multiplier from $\dot{H}^s(\mathbb{R}^d)$ to its dual $\dot{H}^{-s}(\mathbb{R}^d)$. We denote this space of multipliers by $Mult(\dot{H}^s(\mathbb{R}^d) \to \dot{H}^{-s}(\mathbb{R}^d))$.

For $s = 1$, in [11], Maz'ya and Verbitsky proved that $\varphi \in Mult(\dot{H}^1(\mathbb{R}^d) \to \dot{H}^{-1}(\mathbb{R}^d))$ if and only if $(-\Delta)^{-1/2}\varphi \in Mult(\dot{H}^1(\mathbb{R}^d) \to L^2(\mathbb{R}^d))$. Thus, they reduced the characterization of the inequality

$$\left| \int_{\mathbb{R}^d} u(x)v(x)\varphi(x)dx \right| \lesssim \|u\|_{\dot{H}^1(\mathbb{R}^d)} \|v\|_{\dot{H}^1(\mathbb{R}^d)},$$

where φ may change sign, to the inequality

$$\int_{\mathbb{R}^d} |u(x)|^2 |(-\Delta)^{-1/2}\varphi(x)|^2 dx \lesssim \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx,$$

where $|(-\Delta)^{-1/2}\varphi(x)|^2 dx$ is now a non-negative measure. These estimates are of big interest in different aspects of the theory of the Schrödinger operator $(-\Delta + V)$ in $\mathbb{R}^d$. In [12], the same authors proved the non-homogeneous case and an analogous result for $s = 1/2$.

Please cite this article in press as: C. Cascante et al., Bilinear forms on homogeneous Sobolev spaces, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.08.033