J. Math. Anal. Appl. 454 (2017) 585–616

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Well-posedness of stochastic second grade fluids

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A R T I C L E   I N F O

Article history:
Received 22 April 2016
Available online 5 May 2017
Submitted by A. Mazzucato

Keywords:
Second grade fluid
Solvability
Stability
Stochastic

A B S T R A C T

The theory of turbulent Newtonian fluids shows that the choice of the boundary condition is a relevant issue because it can modify the behavior of a fluid by creating or avoiding a strong boundary layer. In this study, we consider stochastic second grade fluids filling a two-dimensional bounded domain with the Navier-slip boundary condition (with friction). We prove the well-posedness of this problem and establish a stability result. Our stochastic model involves a multiplicative white noise and a convective term with third order derivatives, which significantly complicate the analysis.

1. Introduction

The study considers stochastic incompressible fluids of second grade, which are a special class of non-Newtonian fluids. Unlike Newtonian fluids where only the stretching tensor appears in the characterization of the stress response to a deformation fluid, the Cauchy stress tensor $\mathbb{T}$ of non-Newtonian fluids is defined by

$$\mathbb{T} = -\pi\mathbb{I} + \nu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2,$$

where the first term $-\pi\mathbb{I}$ is due to the incompressibility of the fluid and $\mathbb{A}_1$, $\mathbb{A}_2$ are the two first Rivlin–Ericksen tensors (cf. [35])

$$\mathbb{A}_1(y) = \nabla y + (\nabla y)^\top \quad \text{and} \quad \mathbb{A}_2(y) = \dot{\mathbb{A}}_1(y) + \mathbb{A}_1(y)\nabla y + (\nabla y)^\top \mathbb{A}_1(y),$$

where y denotes the velocity of the fluid, the superposed dot is the material time derivative, ν is the kinematic viscosity of the fluid, and α_1, α_2 are constant material moduli. A previous study [18] showed

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http://dx.doi.org/10.1016/j.jmaa.2017.04.060
0022-247X/

that the thermodynamic laws and stability principles impose $\alpha_1 \geq 0$ and $\alpha_1 + \alpha_2 = 0$. We set $\alpha = \alpha_1$ and assume that $\alpha_1 > 0$.

It is well known that small random perturbations in turbulent fluids can produce relevant macroscopic effects. Therefore, the incorporation of stochastic white noise force in the Navier–Stokes equations [3] is widely recognized as important for understanding turbulence phenomena. Thus, in [2] (see Lemma 2.2), the stochastic Navier–Stokes equations were deduced from fundamental principles by showing that the stochastic Navier–Stokes equations are a real physical model. The stochastic Navier–Stokes equations are now quite well understood (e.g., see [16,20,30,36], and the references therein). However, few results have been reported regarding stochastic non-Newtonian fluids [17,32–34]. In this study, we consider the stochastic second grade equations with multiplicative noise given by

$$\begin{cases} \frac{\partial}{\partial t}(Y - \alpha\Delta Y) = \nu\Delta Y - \operatorname{curl}(Y - \alpha\Delta Y) \times Y - \nabla\pi + U + G(t,Y)\,\dot{W}_t, \\ \operatorname{div} Y = 0 \qquad \text{in } \mathcal{O} \times (0,T), \end{cases} \tag{1.1}$$

where U is a body force, $G(t,Y)\,\dot{W}_t$ is a multiplicative white noise, and $\mathcal{O}$ is a bounded domain of $\mathbb{R}^2$ with a boundary Γ.

Studying this system requires suitable boundary conditions on the boundary Γ of the domain. The Dirichlet boundary condition given by

$$Y = 0 \qquad \text{on } \Gamma$$

is accepted as an appropriate boundary condition and it is the most usual. Another physical relevant boundary condition considered in previous studies is the Navier boundary condition

$$Y \cdot \mathrm{n} = 0, \qquad [2(\mathrm{n} \cdot DY) + \gamma Y] \cdot \tau = 0 \qquad \text{on } \Gamma, \tag{1.2}$$

where $\mathrm{n} = (n_1, n_2)$ and $\tau = (-n_2, n_1)$ are the unit normal and tangent vectors, respectively, to the boundary Γ, $DY = \frac{\nabla Y + (\nabla Y)^\top}{2}$ is the symmetric part of the velocity gradient, and $\gamma > 0$ is a friction coefficient on Γ.

The stochastic partial differential equations (1.1) with the Dirichlet boundary condition were studied by [32] and [34]. In the former study, tightness arguments were used together with the Skorohod theorem to prove the existence of a weak stochastic solution in the sense that the Brownian motion, which is part of the solution, was not given in advance; whereas in the second study, the existence and uniqueness of a strong stochastic solution was proved. In pioneering studies [31] and [13] (see also [12]), the deterministic second grade equations with the Dirichlet boundary condition were studied mathematically for the first time, while [6] investigated the deterministic equations with a particular Navier boundary condition (without friction, i.e., when $\gamma = 0$). The physical interpretations of these second grade equations were given by [8,18,19,21,23], and [24]. It is relevant to recall that the deterministic methods are based on the Faedo–Galerkin approximation method and a priori estimates. Then, compactness arguments can be used to pass to the limit of the respective approximate equations in the distributional sense. Unfortunately, for the stochastic partial differential equations, a priori estimates are not sufficient to pass to the limit of the approximate equations due to the lack of regularity on the time and stochastic variables. Thus, in order to obtain a strong stochastic solution, we should verify that the sequence of the Galerkin approximations converges strongly in some adequate topology.

We should note that even if the Dirichlet boundary condition is widely accepted as an appropriate boundary condition at the surface of the contact between a fluid and a solid, it is also a source of many problems because it attaches fluid particles to the boundary, thereby creating a strong boundary layer (cf. [15,25,26,28]). In addition, the Navier boundary condition allows the slippage of the fluid on the boundary, which