J. Math. Anal. Appl. 447 (2017) 867–881

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Symmetries and conservation laws of the Euler equations in Lagrangian coordinates

Ravi Shankar

Department of Mathematics and Statistics, California State University, Chico, CA 95929, USA [1]

ARTICLE INFO

Article history:
Received 2 May 2015
Available online 28 October 2016
Submitted by G. Bluman

Keywords:
Euler equations
Symmetries
Conservation laws
Incompressible flows
Lagrangian coordinates
Time-periodic solution

ABSTRACT

We consider the Euler equations of incompressible inviscid fluid dynamics. We discuss a variational formulation of the governing equations in Lagrangian coordinates. We compute variational symmetries of the action functional and generate corresponding conservation laws in Lagrangian coordinates. We clarify and demonstrate relationships between symmetries and the classical balance laws of energy, linear momentum, center of mass, angular momentum, and the statement of vorticity advection. Using a newly obtained scaling symmetry, we obtain a new conservation law for the Euler equations in Lagrangian coordinates in n-dimensional space. The resulting integral balance relates the total kinetic energy to a new integral quantity defined in Lagrangian coordinates. This relationship implies an inequality which describes the radial deformation of the fluid, and shows the non-existence of time-periodic solutions with nonzero, finite energy.

1. Introduction

This study concerns the Euler equations of incompressible inviscid fluid dynamics. The Euler equations provide an accurate representation of a variety of inviscid fluid flows and are used in numerous practical situations. Mathematical interest in the Euler equations includes the open problems of classifying nonlinear blow-up, long-time behavior, and stability of solutions for various types of initial conditions. Conservation of energy plays an important role in controlling blow-up; for example, solutions of infinite energy have been shown to blow-up in a finite amount of time [9].

The goal of this study is to use variational symmetries of the Euler equations in Lagrangian coordinates to find conservation laws. Noether's First Theorem [19] relates variational symmetries to conservation laws of a corresponding differential system. However, the Euler equations in Eulerian variables do not admit a variational formulation. Point symmetries of the Euler equations in Eulerian variables were obtained in [4]; see also [20]. There are many works on the conservation laws of the Euler equations; see [13] and references

E-mail addresses: rshankar@mail.csuchico.edu, shankarr@uw.edu.

[1] After July 1, 2016: Department of Mathematics, University of Washington, Seattle, WA 98195, USA.

http://dx.doi.org/10.1016/j.jmaa.2016.10.057

therein; see also [1–3] for conservation laws in n-space computed using the direct method. Conservation laws were also obtained for the Euler equations in vorticity formulation [8]. On the other hand, the Euler equations in Lagrangian variables (see [17,10] for background) do admit a variational formulation. Caviglia and Morro [6] computed symmetries and conservation laws using a variational formulation for the compressible Euler equations in Lagrangian coordinates. However, they later [7] computed conservation laws for the incompressible Euler equations without using symmetries or a variational formulation. We show that a simple modification to their compressible flow Lagrangian in [6] allows for a variational formulation of incompressible flow.

In this paper we study a variational formulation of the incompressible Euler equations in Lagrangian coordinates. We obtain a new scaling symmetry of the action functional. This scaling symmetry leads via Noether's theorem to a new conservation law in n-space. The resulting conserved quantity relates the total energy integral, a quantity defined in Eulerian variables, to a new integral quantity defined exclusively in Lagrangian variables. This relationship allows for the quantification of the radial deformation of the fluid, and rules out the existence of time-periodic solutions with nonzero, finite energy.

2. Variational framework

We first introduce Lagrangian coordinates, discuss their physical interpretation, and detail explicitly the transformation of the Euler equations to Lagrangian coordinates from Eulerian variables. Then, we demonstrate the variational formulation behind the Euler equations in Lagrangian coordinates using a Lagrange multiplier (namely the pressure) of the incompressibility constraint.

2.1. Lagrangian coordinates

The continuity and Euler equations of incompressible ideal fluid dynamics are as follows [16]:

$$\vec{\nabla}_x \cdot \vec{u} = 0, \tag{1}$$

$$(\partial_t + \vec{u} \cdot \vec{\nabla}_x)\vec{u} = -\vec{\nabla}_x p, \tag{2}$$

where $\vec{u} = (u^1, \ldots, u^n)$ is the velocity vector, $\vec{\nabla}_x = (\partial_{x^1}, \ldots, \partial_{x^n})$ is the gradient, and $p = \bar{p}/\rho$ is the pressure $\bar{p}$ divided by the constant density ρ; we henceforth refer to p as the pressure. The velocity vector $\vec{u}$ and pressure p are functions of $(\vec{x}, t) = (x^1, \ldots, x^n, t)$. The first condition (1) gives the local conservation of mass and specifies that the fluid be incompressible. The second states the momentum balance within a small parcel of fluid. The equations are considered on $\mathbb{R}^n \times [0, \infty)$, where $n \geq 2$ is 2 or 3 in most applications.

The equations for ideal incompressible fluid dynamics can be recast using the "Lagrangian map" $\vec{x}(\vec{a}, t) = (x^1, \ldots, x^n)$ [15,5]. This smooth function gives the position $\vec{x}$ of a particular fluid particle given by "label vector" $\vec{a} = (a^1, \ldots, a^n)$ at time t. At the initial time $t = 0$, the positions of the fluid particles coincide with their "Lagrangian coordinates": $\vec{x}(\vec{a}, 0) = \vec{a}$. As time progresses, the positions of the particles deviate from their initial positions by the action of the fluid velocity $\vec{u}(\vec{x}, t)$. This relationship between position and velocity can be quantified by a kinematic system of equations: $\partial_t \vec{x}(\vec{a}, t) = \vec{u}(\vec{x}, t)$, where ∂_t denotes partial differentiation with respect to variable t. For small times $t \to 0$, Taylor's theorem gives

$$\begin{aligned} \vec{x}(\vec{a}, t) &\approx \vec{x}(\vec{a}, 0) + t\ \partial_t \vec{x}(\vec{a}, 0) + O(t^2) \\ &= \vec{a} + t\, \vec{u}(\vec{x}(\vec{a}, 0), 0) + O(t^2) \\ &= \vec{a} + t\, \vec{u}(\vec{a}, 0) + O(t^2). \end{aligned} \tag{3}$$

In this moving frame that follows the fluid particles, governing equations (1), (2) take the following form [17,10]: