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Phase retrieval from the magnitudes of affine linear measurements

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ABSTRACT

In this paper, we consider the affine phase retrieval problem in which one aims to recover a signal from the magnitudes of affine measurements. Let $\{\mathbf{a}_j\}_{j=1}^m \subset \mathbb{H}^d$ and $\mathbf{b} = (b_1, \ldots, b_m)^\top \in \mathbb{H}^m$, where $\mathbb{H} = \mathbb{R}$ or $\mathbb{C}$. We say $\{\mathbf{a}_j\}_{j=1}^m$ and $\mathbf{b}$ are affine phase retrievable for $\mathbb{H}^d$ if any $\mathbf{x} \in \mathbb{H}^d$ can be recovered from the magnitudes of the affine measurements $\{|\langle \mathbf{a}_j, \mathbf{x}\rangle + b_j|,\ 1 \le j \le m\}$. We develop general framework for affine phase retrieval and prove necessary and sufficient conditions for $\{\mathbf{a}_j\}_{j=1}^m$ and $\mathbf{b}$ to be affine phase retrievable. We establish results on minimal measurements and generic measurements for affine phase retrieval as well as on sparse affine phase retrieval. In particular, we also highlight some notable differences between affine phase retrieval and the standard phase retrieval in which one aims to recover a signal $\mathbf{x}$ from the magnitudes of its linear measurements. In standard phase retrieval, one can only recover $\mathbf{x}$ up to a unimodular constant,

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while affine phase retrieval removes this ambiguity. We prove that unlike standard phase retrieval, the affine phase retrievable measurements $\{\mathbf{a}_j\}_{j=1}^m$ and $\mathbf{b}$ do not form an open set in $\mathbb{H}^{m\times d}\times\mathbb{H}^m$. Also in the complex setting, the standard phase retrieval requires $4d-O(\log_2 d)$ measurements, while the affine phase retrieval only needs $m = 3d$ measurements.

1. Introduction

1.1. Phase retrieval

Phase retrieval is an active topic of research in recent years as it arises in many different areas of studies (see [2,5,6,9–12,15] and the references therein). For a vector (signal) $\mathbf{x}\in\mathbb{H}^d$, where $\mathbb{H}=\mathbb{R}$ or $\mathbb{C}$, the aim of phase retrieval is to recover $\mathbf{x}$ from $|\langle\mathbf{a}_j,\mathbf{x}\rangle|$, $j=1,\dots,m$, where $\mathbf{a}_j\in\mathbb{H}^d$ and we usually refer to $\{\mathbf{a}_j\}_{j=1}^m$ as the *measurement vectors*. Since for any unimodular $c\in\mathbb{H}$, we have $|\langle\mathbf{a}_j,\mathbf{x}\rangle|=|\langle\mathbf{a}_j,c\mathbf{x}\rangle|$, the best outcome phase retrieval can achieve is to recover $\mathbf{x}$ up to a unimodular constant.

We briefly overview some of the results in phase retrieval and introduce some notations. For the set of measurement vectors $\{\mathbf{a}_j\}_{j=1}^m$, we set $\mathbf{A}:=(\mathbf{a}_1,\dots,\mathbf{a}_m)^\top\in\mathbb{H}^{m\times d}$ which we shall refer to as the *measurement matrix*. We shall in general identify the set of measurement vectors $\{\mathbf{a}_j\}_{j=1}^m$ with the corresponding measurement matrix $\mathbf{A}$, and often use the two terms interchangeably whenever there is no confusion. Define the map $\mathbf{M}_\mathbf{A}:\mathbb{H}^d\to\mathbb{R}^m_{\geq 0}$ by

$$\mathbf{M}_\mathbf{A}(\mathbf{x}) \;=\; (|\langle\mathbf{a}_1,\mathbf{x}\rangle|,\dots,|\langle\mathbf{a}_m,\mathbf{x}\rangle|).$$

We say $\mathbf{A}$ is *phase retrievable* for $\mathbb{H}^d$ if $\mathbf{M}_\mathbf{A}(\mathbf{x})=\mathbf{M}_\mathbf{A}(\mathbf{y})$ implies $\mathbf{x}\in\{c\mathbf{y}: c\in\mathbb{H}, |c|=1\}$. There have been extensive studies of phase retrieval from various different angles. For example many efficient algorithms to recover $\mathbf{x}$ from $\mathbf{M}_\mathbf{A}(\mathbf{x})$ have been developed, see [7–9,17] and their references. One of the fundamental problems on the theoretical side of phase retrieval is the following question: *How many vectors in the measurement matrix* $\mathbf{A}$ *are needed so that* $\mathbf{A}$ *is phase retrievable?* It is shown in [2] that for $\mathbf{A}$ to be phase retrievable for $\mathbb{R}^d$, it is necessary and sufficient that $m\geq 2d-1$.

In the complex case $\mathbb{H}=\mathbb{C}$, the same question becomes much more challenging, however. The minimality question remains open today. Balan, Casazza and Edidin [2] first show that $\mathbf{A}$ is phase retrievable if it contains $m\geq 4d-2$ generic vectors in $\mathbb{C}^d$. Bodmann and Hammen [5] show that $m=4d-4$ measurement vectors are possible for phase retrieval through construction (see also Fickus, Mixon, Nelson and Wang [12]). Bandeira, Cahill, Mixon and Nelson [4] conjecture that (a) $m\geq 4d-4$ is necessary for $\mathbf{A}$ to be phase retrievable and, (b) $\mathbf{A}$ with $m\geq 4d-4$ generic measurement vectors is phase retrievable. Part (b) of the conjecture is proved by Conca, Edidin, Hering and Vinzant [11].