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Profinite automata

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A B S T R A C T

Many sequences of p-adic integers project modulo p^α to p-automatic sequences for every $\alpha \geq 0$. Examples include algebraic sequences of integers, which satisfy this property for every prime p, and some cocycle sequences, which we show satisfy this property for a fixed p. For such a sequence, we construct a profinite automaton that projects modulo p^α to the automaton generating the projected sequence. In general, the profinite automaton has infinitely many states. Additionally, we consider the closure of the orbit, under the shift map, of the p-adic integer sequence, defining a shift dynamical system. We describe how this shift is a letter-to-letter coding of a shift generated by a constant-length substitution defined on an uncountable alphabet, and we establish some dynamical properties of these shifts.

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1. Introduction

A *substitution* (or *morphism*) on an alphabet $\mathcal{A}$ is a map $\theta : \mathcal{A} \to \mathcal{A}^*$, extended to $\mathcal{A}^{\mathbb{N}}$ by concatenation. A substitution is *length-k* (or *k-uniform*) if, for each $a \in \mathcal{A}$, the length of $\theta(a)$ is k. The extensive literature on substitutions has traditionally focused on the case where $\mathcal{A}$ is finite. Some exceptions include recent work, for example in [12] and [14]. Substitutions on a countably infinite alphabet have been used to describe lexicographically least sequences on $\mathbb{N}$ avoiding certain patterns [13,19], and they have been used in the combinatorics literature to enumerate permutations avoiding patterns [25].

In this article we present a new construction for constant-length substitutions on an uncountable alphabet. Our motivation comes from the following classical results. Let $(a(n))_{n\geq 0}$ be an *automatic sequence* (see Definition 2.2). Cobham's theorem (Theorem 2.6) characterizes an automatic sequence as the coding, under a letter-to-letter map, of a fixed point of a constant-length substitution. Christol's theorem (Theorem 2.8) characterizes p-automatic sequences for prime p; they are precisely the sequences whose generating function is algebraic over a finite field of characteristic p. The following can be viewed as a generalization of one direction of Christol's characterization.

Theorem 1.1 *([6, Theorem 32], [10, Theorem 3.1]). Let $(a(n))_{n\geq 0}$ be a sequence of p-adic integers such that $\sum_{n\geq 0} a(n)x^n$ is algebraic over $\mathbb{Z}_p(x)$, and let $\alpha \geq 0$. Then $(a(n) \bmod p^\alpha)_{n\geq 0}$ is p-automatic.*

Thus certain p-adic integer sequences (and, in particular, integer sequences) have the property that they become p-automatic when reduced modulo p^α, for every $\alpha \geq 0$. More generally, the diagonal of a multivariate rational power series is p-automatic when reduced modulo p^α, and one can explicitly compute an automaton for $(a(n) \bmod p^\alpha)_{n\geq 0}$ for all but finitely many primes p [20, Theorem 2.1].

Fix a prime p, and let $(a(n))_{n\geq 0}$ be a sequence such that $(a(n) \bmod p^\alpha)_{n\geq 0}$ is p-automatic for every $\alpha \geq 0$. For each α, there is a finite automaton generating $(a(n) \bmod p^\alpha)_{n\geq 0}$. In Lemma 3.1 we show that these automata can be chosen in a compatible way; namely, their inverse limit exists. In this way we obtain a *profinite automaton* (Definition 3.3) generating the sequence $(a(n))_{n\geq 0}$.

We can obtain other inverse limit objects from a p-adic integer sequence in a similar way. In particular, Cobham's theorem guarantees a length-p substitution θ_α such that $(a(n) \bmod p^\alpha)_{n\geq 0}$ is a coding of a fixed point of θ_α. Each substitution θ_α is a substitution on a finite alphabet, but their inverse limit is a profinite substitution on an alphabet that is, in general, uncountable (Theorem 4.2). This alphabet has a natural coding to the set $\mathbb{Z}_p$ of p-adic integers, and the sequence $(a(n))_{n\geq 0}$ is the coding of a fixed point of the profinite substitution.

With this profinite substitution, we obtain a shift (Theorem 4.1), as in the classical finite-alphabet case. This shift is the closure of the orbit, under the shift map, of a fixed point (or coding of a fixed point) of the profinite substitution. One feature of profinite

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