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Flag-transitive quasi-residual designs with sporadic socle☆

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ABSTRACT

Let $\mathcal{D}$ be a quasi-residual design with an automorphism group G. In this paper, we classify flag-transitive, block-primitive or point-primitive quasi-residual designs with sporadic socle. Furthermore, we show that if G is flag-transitive block-primitive with sporadic socle, then G is point-primitive.

1. Introduction

A 2-(v, k, λ) *design* $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is an incidence structure consisting of a finite set $\mathcal{P}$ of v points and a collection $\mathcal{B}$ of some distinct k-subsets of $\mathcal{P}$, called blocks, such that each 2-subset of $\mathcal{P}$ is contained in precisely λ blocks. We denote the number of blocks by b and the number of blocks containing a given point by r. The numbers v, b, r, k, λ are the parameters of $\mathcal{D}$. A design is called *symmetric* if $b = v$ (or equivalently $r = k$), and *non-trivial* if $2 < k < v$. A *residual design* of a symmetric design $\mathcal{D}$ is obtained by removing a block B of $\mathcal{B}$ and all points in B from the other blocks. From the definition we know that a 2-(v, k, λ) residual design satisfies $r = k + \lambda$. A 2-(v, k, λ) design is called a *quasi-residual design* if $r = k + \lambda$ (or equivalently $b = v + r - 1$).

A *flag* in $\mathcal{D}$ is an incidence point-block pair. The set of all flags in $\mathcal{D}$ is denoted by $\mathcal{F}$. For a point $x \in \mathcal{P}$, denote $\mathcal{P}(x)$ the set of all blocks which are incident with x. Let $G \leq Aut(\mathcal{D})$ be an automorphism group of a 2-(v, k, λ) design $\mathcal{D}$. Then G is called *point-transitive* (*block-transitive*, *flag-transitive*, respectively) on $\mathcal{D}$ if G is transitive on $\mathcal{P}$ ($\mathcal{B}$, $\mathcal{F}$, respectively); G is called *point-primitive* (*block-primitive*, respectively) on $\mathcal{D}$ if G is primitive on $\mathcal{P}$ ($\mathcal{B}$, respectively). A set of blocks is called *base blocks* with respect to an automorphism group G of design $\mathcal{D}$ if it contains exactly one block from each G-orbit on the block set. In particular, any block is a base block of $\mathcal{D}$ if G is a block-transitive automorphism group of $\mathcal{D}$.

In 2005, Tian and Zhou completely classified flag-transitive point-primitive symmetric designs with sporadic socle [12]; in 2016, Zhan and Zhou dealt with flag-transitive nonsymmetric designs with $(r, \lambda) = 1$ and sporadic socle [13]. In [8], Delandtsheer had given a conjecture: if $G \leq Aut(\mathcal{D})$ acts block-primitively on a 2-$(v, k, 1)$ design $\mathcal{D}$, then G is point-primitive. Although there are many facts supporting this conjecture [7,8,10,14], but it is still open.

In this paper, we give the classification of flag-transitive, block-primitive or point-primitive quasi-residual 2-(v, k, λ) designs with sporadic socle. Furthermore, as a corollary, for a flag-transitive, quasi-residual 2-(v, k, λ) design with sporadic socle, we show that if $G \leq Aut(\mathcal{D})$ is block-primitive, then G is also point-primitive. In 2000, Camina and Spiezia proved that if G is a group acting block-transitively on a 2-$(v, k, 1)$ design, then the socle of G is not a sporadic simple group [3]. In addition, if $k = v - 1$, then $r = v - 1$, which means that $\mathcal{D}$ is not a quasi-residual design. Thus, we only consider that $\lambda > 1$ and $2 < k < v - 1$.

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Table 1
Block-primitive quasi-residual designs.

Case	G	G_B	(v, b, r, k, λ)	$\mathcal{D}$
1	M_{11}	S_5	(12, 66, 55, 10, 45)	$\mathcal{D}_1$
2	M_{12}	M_{10}: 2	(12, 66, 55, 10, 45)	$\mathcal{D}_2, \mathcal{D}_3 \cong \mathcal{D}_1$
3	M_{22}	2^4: A_6	(22, 77, 56, 16, 40)	$\mathcal{D}_4$
4		2^4: S_5	(22, 231, 210, 20, 190)	$\mathcal{D}_5$
5	M_{22}: 2	2^4: S_6	(22, 77, 56, 16, 40)	$\mathcal{D}_6 \cong \mathcal{D}_4$
6		2^5: S_5	(22, 231, 210, 20, 190)	$\mathcal{D}_7 \cong \mathcal{D}_5$
7	M_{23}	$L_3(4)$: 2_2	(23, 253, 231, 21, 210)	$\mathcal{D}_8$
8	M_{24}	M_{22}: 2	(24, 276, 253, 22, 231)	$\mathcal{D}_9$

Table 2
Point-primitive quasi-residual designs.

Case	G	G_x	(v, b, r, k, λ)	$\mathcal{D}$
1	M_{11}	M_{10}	(11, 55, 45, 9, 36)	$\mathcal{D}_{10}$
2		$L_2(11)$	(12, 22, 11, 6, 5)	$\mathcal{D}_{11} \cong \mathcal{D}_1$
3		$L_2(11)$	(12, 66, 55, 10, 45)	$\mathcal{D}_{12}$
4	M_{12}	M_{11}	(12, 66, 55, 10, 45)	$\mathcal{D}_{13} \cong \mathcal{D}_{12} \cong \mathcal{D}_2$
5	M_{22}	$L_3(4)$	(22, 77, 56, 16, 40)	$\mathcal{D}_{14} \cong \mathcal{D}_4$
6		$L_3(4)$	(22, 231, 210, 20, 190)	$\mathcal{D}_{15} \cong \mathcal{D}_5$
7	M_{22}: 2	$L_3(4)$: 2_2	(22, 77, 56, 16, 40)	$\mathcal{D}_{16} \cong \mathcal{D}_{14} \cong \mathcal{D}_6$
8		$L_3(4)$: 2_2	(22, 231, 210, 20, 190)	$\mathcal{D}_{17} \cong \mathcal{D}_{15} \cong \mathcal{D}_7$
9	M_{23}	M_{22}	(23, 253, 231, 21, 210)	$\mathcal{D}_{18} \cong \mathcal{D}_8$
10	M_{24}	M_{23}	(24, 276, 253, 22, 231)	$\mathcal{D}_{19} \cong \mathcal{D}_9$

Theorem 1.1. *Let $\mathcal{D}$ be a non-trivial quasi-residual 2-(v, k, λ) design with a flag-transitive, block-primitive automorphism group G whose socle is a sporadic simple group. Then one of the following holds:*

(1) *$\mathcal{D}$ is a unique 2-(12, 10, 45) design, $G = M_{11}$ or $G = M_{12}$;*
(2) *$\mathcal{D}$ is a 2-(22, 16, 40) or a 2-(22, 20, 190) design, $G = M_{22}$ or $G = M_{22} : 2$;*
(3) *$\mathcal{D}$ is a unique 2-(23, 21, 210) design, $G = M_{23}$;*
(4) *$\mathcal{D}$ is a unique 2-(24, 22, 231) design, $G = M_{24}$.*

Remark 1. G and $\mathcal{D}$ can be read from Table 1. There are five different designs up to isomorphism.

Theorem 1.2. *Let $\mathcal{D}$ be a non-trivial quasi-residual 2-(v, k, λ) design with a flag-transitive, point-primitive automorphism group G whose socle is a sporadic simple group. Then one of the following holds:*

(1) *$\mathcal{D}$ is a unique 2-(11, 9, 36) or 2-(12, 6, 5) design, $G = M_{11}$;*
(2) *$\mathcal{D}$ is a unique 2-(12, 10, 45) design, $G = M_{11}$ or $G = M_{12}$;*
(3) *$\mathcal{D}$ is a 2-(22, 16, 40) or a 2-(22, 20, 190) design, $G = M_{22}$ or $G = M_{22} : 2$;*
(4) *$\mathcal{D}$ is a unique 2-(23, 21, 210) design, $G = M_{23}$;*
(5) *$\mathcal{D}$ is a unique 2-(24, 22, 231) design, $G = M_{24}$.*

Remark 2. G and $\mathcal{D}$ can be read from Table 2. There are seven different designs up to isomorphism. All pairs $(G, \mathcal{D})$ in Theorem 1.1 also appear in Theorem 1.2, and all designs can be found in [4].

Theorem 1.3. *Let $\mathcal{D}$ be a non-trivial quasi-residual 2-(v, k, λ) design with a flag-transitive automorphism group G whose socle is a sporadic simple group. If G is block-primitive, then G is point-primitive.*

2. Preliminaries

In this section, we state some preliminary results which will be needed in this paper.

Lemma 2.1 ([4])**.** *Let $\mathcal{D}$ be a 2-(v, k, λ) design. Then the following holds:*

(1) $r(k-1) = \lambda(v-1)$;
(2) $bk = vr$;
(3) $b \geq v$.

Lemma 2.2. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a non-trivial 2-(v, k, λ) design with an automorphism group G, $x \in \mathcal{P}$, $B \in \mathcal{B}$. Then G is flag-transitive if and only if G is point-transitive and G_x is transitive on $\mathcal{P}(x)$; or G is block-transitive and G_B is transitive on B.*

Lemma 2.3. *Let $\mathcal{D}$ be a 2-(v, k, λ) design admitting a flag-transitive automorphism group G. Then the following holds:*

(1) *$r | \lambda d$, where d is any non-trivial subdegree of G;*