Applied Mathematics and Computation 316 (2018) 480–487

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A lower bound of revised Szeged index of bicyclic graphs

Shengjin Ji [a,c], Mengmeng Liu [b,*], Jianliang Wu [c]

[a] School of Science, Shandong University of Technology, Zibo, Shandong 255049, China
[b] School of Mathematics, Lanzhou Jiaotong University, Lanzhou, Gansu 730070, China
[c] School of Mathematics, Shandong University, Jinan, Shandong 250100, China

ARTICLE INFO

Keywords:
Wiener index
Revised Szeged index
Bicyclic graph

ABSTRACT

The revised Szeged index of a graph is defined as $Sz^*(G) = \sum_{e=uv\in E}(n_u(e) + \frac{n_0(e)}{2})(n_v(e) + \frac{n_0(e)}{2})$, where $n_u(e)$ and $n_v(e)$ are, respectively, the number of vertices of G lying closer to vertex u than to vertex v and the number of vertices of G lying closer to vertex v than to vertex u, and $n_0(e)$ is the number of vertices equidistant to u and v. In the paper, we identify the lower bound of revised Szeged index among all bicyclic graphs, and also characterize the extremal graphs that attain the lower bound.

1. Introduction

A map taking graphs as arguments is referred to as a graph invariant if it assigns equal values to isomorphic graphs. These invariants have been used for modeling some properties of chemical compounds and capturing the structural essence of compounds with respect to a molecule, which, in chemical graph theory, are also called the topological indices.

They include graph energy, various of graph like-energies, Randić index, Zagreb index, PI index and graph entropies, etc, see literatures [3,6,11,16,18,19,23,25,28–31,37,39,40] and cited in them for properties and applications of the variants.

All graph considered in this paper are finite, undirected and simple. We refer the readers to [5] for terminology and notation. The *Wiener index* of G is defined as:

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v),$$

where $d(u,v)$ is the distance between u and v in G. This topological index has been extensively studied, see [12,14,15,17,22,42]. Let $e = uv$ be an edge of G, and define three subsets of $V(G)$ as follows:

$$\begin{aligned}
N_u(e) &= \{w \in V(G) : d(u,w) < d(v,w)\},\\
N_v(e) &= \{w \in V(G) : d(u,w) > d(v,w)\},\\
N_0(e) &= \{w \in V(G) : d(u,w) = d(v,w)\}.
\end{aligned}$$

In fact, $\{N_u(e), N_v(e), N_0(e)\}$ consists of a partition of vertices set $V(G)$ with respect to e. The number of vertices of $N_u(e), N_v(e), N_0(e)$ are denoted by $n_u(e), n_v(e), n_0(e)$, respectively. Evidently, if n is the number of vertices of the graph G, then $n_u(e) + n_v(e) + n_0(e) = n$.

* Corresponding author.
E-mail addresses: jishengjin2013@163.com (S. Ji), liumm05@163.com, cs_29@126.com (M. Liu), jlwu@sdu.edu.cn (J. Wu).

http://dx.doi.org/10.1016/j.amc.2017.08.051
0096-3003/

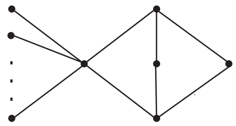

Fig. 1. Graph B used in the proof of Theorem 1.

The Wiener index $W(T)$ of a tree T can be computed as follows:

$$W(T) = \sum_{e=uv\in E(T)} n_u(e)n_v(e).$$

For extending the above formula to general graph G, Gutman [13] introduced a graph invariant named the *Szeged index* and defined it by

$$Sz(G) = \sum_{e=uv\in E(G)} n_u(e)n_v(e).$$

Randić [36] observed that the Szeged index does not take into account the contributions of the vertices at equal distances from the endpoints of an edge. He thus conceived a modified version of the Szeged index and named the *revised Szeged index*. The revised Szedged index of a connected graph G is defined as

$$Sz^*(G) = \sum_{e=uv\in E(G)} \left(n_u(e) + \frac{n_0(e)}{2}\right)\left(n_v(e) + \frac{n_0(e)}{2}\right).$$

It is well known that for a connected graph G, $Sz^*(G) \geq Sz(G) \geq W(G)$. It is natural to ask the following question: What is maximum (minimum) of the differences $Sz(G) - W(G)$ and $Sz^*(G) - W(G)$, and which graph meets the maximal (minimum) value? This question has attracted many mathematicians to focus, see [4,7,9,10,26,32,33,44–46] for details.

In addition, some properties and applications of these two topological indices have been reported in [2,8,20,21,24,34,35,38,41]. Aouchiche and Hansen [1] showed that for a connected graph G of order n and size m, an upper bound of the revised Szeged index of G is $\frac{n^2m}{4}$. In [43], Xing and Zhou acquired the unicyclic graphs of order n with the smallest and largest revised Szeged indices for $n \geq 5$. Li and Liu [27] identified those graphs whose revised Szeged index is maximal among bicyclic graphs. In this paper, we give a lower bound of the revised Szeged index for a connected bicyclic graph and also characterize those graphs that achieve the lower bound.

Theorem 1. *Let G be a connected bicyclic graph of order $n \geq 17$. Then*

$$Sz^*(G) \geq n^2 + 8n - 29.$$

with equality if and only if $G \cong B$ (see Fig. 1).

2. Preliminaries

From the fact that $n_u(e) + n_v(e) + n_0(e) = n$ and $m = n + 1$, we have

$$\begin{aligned} Sz^*(G) &= \sum_{e=uv\in E} \left(n_u(e) + \frac{n_0(e)}{2}\right)\left(n_v(e) + \frac{n_0(e)}{2}\right) \\ &= \sum_{e=uv\in E} \left(\frac{n + n_u(e) - n_v(e)}{2}\right)\left(\frac{n - n_u(e) + n_v(e)}{2}\right) \\ &= \sum_{e=uv\in E} \frac{n^2 - (n_u(e) - n_v(e))^2}{4} \\ &= \frac{mn^2}{4} - \frac{1}{4}\sum_{e=uv\in E} (n_u(e) - n_v(e))^2. \\ &= \frac{n^3 + n^2}{4} - \frac{1}{4}\sum_{e=uv\in E} (n_u(e) - n_v(e))^2. \end{aligned}$$

For convenience, let $\delta(e) = |n_u(e) - n_v(e)|$, where $e = uv$. we have

$$Sz^*(G) = \frac{n^3 + n^2}{4} - \frac{1}{4}\sum_{e\in E} \delta(e)^2. \tag{1}$$