Applied Mathematics and Computation 313 (2017) 180–195

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Solving over-constrained systems of non-linear interval equations – And its robotic application

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ARTICLE INFO

MSC:
65G40
93E24
70B15

Keywords:
Interval analysis
Resolution
Over-constrained system
Least squares

ABSTRACT

This paper presents and describes in details an original method developed to solve over-constrained systems of non-linear interval equations that arise namely in parameter identification problems deriving from physical models and uncertain measurements. Our approach consists of computing an interval enclosure of the least square solution set and an inner box of tolerable solution set. This method is applied in a detailed example and some interesting results obtained for the calibration of a cable-driven robot are shown.

1. Motivations

In order to improve the control or the global accuracy of a robot, the behavior of a mechanism or just the realism of a physical model, a good knowledge of its parameters needs to be reached. One method to obtain the actual parameters of a model is to identify them through measurements of the state of the phenomenon which is described by the model. In practice, the measurements obtained are noisy and marred by uncertainties. To minimize the effect of this noise on the identification results, it is necessary to multiply the measurements, which leads to averaging the measurement errors. Finally, the parameter identification process amounts to solve an over-constrained system of equations, which are often non-linear, under uncertainties.

The classical way to solve this kind of over-constrained problems is to use the least square based methods [1]. These methods are based on finding the solution minimizing the residual errors on equations. Unfortunately, these methods require a knowledge of the uncertainties distribution and do not propose any certification on the results. The intervals, with a set approach, allow one to represent uncertainties [2], at the only and simple condition that we can bound them. This interval based approach permits to propose different characterizations of the solutions and a certification of the results [3].

In this paper, we firstly present the state of the art in term of guaranteed solving of an over-constrained system of non-linear equations. Next we describe our innovative method which is applied to a basic example in the third section. The last section contains a more serious application, leading to a useful result in the robotic field.

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http://dx.doi.org/10.1016/j.amc.2017.05.077
0096-3003/

2. Interval notation

Herein, we introduce some basic interval concepts and notation. An interval matrix is defined as a set

$$\boldsymbol{A} := \{A \in \mathbb{R}^{m\times n} \mid \underline{A} \le A \le \overline{A}\},$$

where $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$, $\underline{A} \le \overline{A}$, are given matrices and the inequality is understood entry-wise. The corresponding midpoint and radius matrices are respectively defined as

$$A^c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A^\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

The set of all interval $m \times n$ matrices is denoted by $\mathbb{IR}^{m\times n}$.

Intervals and interval vectors are considered as special cases of interval matrices. For interval arithmetic, we refer the readers, e.g., to books [2,4,5].

Let $S \subset \mathbb{R}^n$. An enclosure of the set S is any interval vector $\boldsymbol{x} \in \mathbb{IR}^n$, called often a box, such that $S \subseteq \boldsymbol{x}$. In contrast, an inner estimation of S is any interval vector $\boldsymbol{y} \in \mathbb{IR}^n$ such that $\boldsymbol{y} \subseteq S$.

The smallest interval enclosure of S with respect to inclusion is called the interval hull of S and denoted by $\square S$; formally,

$$\square S := \bigcap_{\boldsymbol{x}\in\mathbb{IR}^n : S\subseteq \boldsymbol{x}} \boldsymbol{x}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{IR}^n$. Then the image of $\boldsymbol{x}$ under the function f is defined as

$$f(\boldsymbol{x}) := \{f(x) \mid x \in \boldsymbol{x}\}.$$

A mapping $\boldsymbol{f} : \mathbb{IR}^n \to \mathbb{IR}$ is called an interval enclosing extension of f if $f(\boldsymbol{x}) \subseteq \boldsymbol{f}(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathbb{IR}^n$. Provided f is given by an arithmetic expression, then evaluation by interval arithmetic is an example of an interval enclosing extension of f.

3. State of the art

In this paper, we focus on an over-constrained system of non-linear equations with the constraint functions of the form

$$f(x, \boldsymbol{y}), \ x \in \mathbb{R} \text{ and } \boldsymbol{y} \in \mathbb{IR} \tag{1}$$

and we propose a method to characterize the solutions x of such a system in the sense $0 \in f(x, \boldsymbol{y})$.

The problem of solving an over-constrained system of non-linear equations with interval coefficients mainly appears in the case of parameter identification in uncertain environment (with x the parameter to identify and $\boldsymbol{y}$ the set of measurements), thus the existing methods come from this field of research. To find the set $\mathcal{X} = \{x \mid \exists y \in \boldsymbol{y}, f(x, y) = 0\}$ is a common problem statement. However, if the model is too far from the reality, it can appear that $\mathcal{X} = \emptyset$. If we consider that the measurements are outliers, i.e., completely wrong, a method based on relaxed intersection can be used [6]. In this paper, we consider the case where measurements can be wrong for the model, but correct w.r.t. the reality. We then want to use it, all the same, to improve the correspondence between the model and the real system. A paving approach is proposed in [7] to bound the parameters of a model by considering many measurements (the example chosen is related to electrochemistry). This constraint programming based method allows to add a robustification to outliers. A method using global optimization to compute an interval based least squares is discussed in [8], the author concludes that a direct solution of the over-constrained system is a more interesting approach than an optimization procedure. However, if a constraint programming based method can manage with more equations than unknowns, the linear algebra based methods need to be adapted. An interval Newton scheme, in which the over-constrained linearized system is solved with a simplex algorithm permits the calibration of a parallel manipulator in [3].

In this paper, we propose an interval Newton scheme based method, in which the over-constrained linearized system is rewritten in a well determined linear system before being solved in two different manners to propose two different solution characterizations. We review the known results first.

Real case. Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $m > n$, and consider the over-constrained system of linear equations $Ax = b$. Since it is unlikely to be solvable, we usually seek for an approximate solution be the least square method. A least square solution is any optimal solution of

$$\min_{s\in\mathbb{R}^n} \|As - b\|_2.$$

A least square solution always exists, and provided A has rank n, then it is unique. It is well-known that least square solutions can be equivalently obtained by solving the normal equations

$$(A^T A)s = A^T b. \tag{2}$$