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A robust numerical method for a fractional differential equation

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ABSTRACT

This paper is devoted to giving a rigorous numerical analysis for a fractional differential equation with order $\alpha \in (0, 1)$. First the fractional differential equation is transformed into an equivalent Volterra integral equation of the second kind with a weakly singular kernel. Based on the *a priori* information about the exact solution, an integral discretization scheme on an *a priori* chosen adapted mesh is proposed. By applying the truncation error estimate techniques and a discrete analogue of Gronwall's inequality, it is proved that the numerical method is first-order convergent in the discrete maximum norm. Numerical results indicate that this method is more accurate and robust than finite difference methods when α is close to 0.

1. Introduction

Fractional calculus has played a significant role in a variety of scientific and engineering fields, such as finance [12,18], control [11], viscoelasticity [6], and hydrology [1,13]. Those fractional models, described in the form of fractional differential equations, have been proved to be more appropriate for modeling many physical phenomena than the traditional integer order models, because fractional calculus enables the description of the memory properties of various materials and processes [17, Chapter 10]. The analytical solutions of most fractional differential equations can not be obtained, so approximate and numerical techniques must be used.

In this paper, we consider the following fractional differential equation

$$D_*^\alpha u(x) + f(x, u) = 0, \quad x \in \Omega := (0, 1], \tag{1.1}$$

$$u(0) = \gamma, \tag{1.2}$$

where D_*^α denotes the Caputo fractional derivative defined by

$$D_*^\alpha u(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} u'(t) dt, \quad 0 < \alpha < 1, \tag{1.3}$$

the constant γ and function f are given. We assume that function f satisfies

$$f(x, u) \in C\left(\bar{\Omega} \times \mathbb{R}\right) \cap C^1(\Omega \times \mathbb{R}). \tag{1.4}$$

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Under this hypothesis, the fractional differential equation (1.1) and (1.2) exists a unique solution $u(x) \in C(\bar{\Omega})$ (see e.g., [3, Theorem 6.8]). In [3, Theorem 6.28] it is shown that if $f(x,u) \in C\left(\bar{\Omega} \times \mathbb{R}\right) \cap C^q(\Omega \times \mathbb{R})$ for some integer $q \geq 1$ there exists a positive constant C_0 such that

$$\left|u^{(j)}(x)\right| \leq \begin{cases} C_0, & \text{if } j=0, \\ C_0 x^{\alpha-j}, & \text{if } j=1,2,\ldots,q+1, \end{cases} \tag{1.5}$$

for all $x \in \Omega$. These bounds indicate that the derivatives of the solution u may blow up at the end point $x=0$. This singular behavior complicates the construction of the discretization scheme and the convergence analysis of the numerical method.

Since analytic solutions for fractional differential equations are generally impossible to attain, various numerical methods for solving the fractional differential equations have been proposed. There are some papers which take account of the possibly singular behavior of solutions of the fractional differential equations. Numerical schemes based on the collocation methods are developed in [4,7–10,14–16], finite difference schemes are proposed in [2,5,19,20]. Because of the presence of the singular behavior, standard numerical methods fail to give accurate approximation, even for high-order methods.

In this paper we first transform the fractional order initial value problem (1.1) and (1.2) into an equivalent Volterra integral equation of the second kind with a weakly singular kernel. Then we discretize the Volterra integral equation on an *apriori* chosen adapted mesh. A rigorous analysis about the convergence of the discretization scheme is given by taking account of the possibly singular behavior of the solution. Applying the truncation error estimate techniques and a discrete analogue of Gronwall's inequality [21], we will prove that the scheme is first-order convergent in the discrete maximum norm. Numerical results are given to display that this method is more accurate and robust than finite difference methods when α is close to 0.

The rest of the paper is organized as follows. The discretization scheme is described in Section 2. Convergence analysis of the scheme is given in Section 3. Finally, numerical experiments are presented in Section 4.

Notation. Throughout the paper, C will denote a generic positive constant that is independent of the mesh. Note that C is not necessarily the same at each occurrence. To simplify the notation we set $g_i = g(x_i)$ for any function $g \in C(\bar{\Omega})$.

2. Discretization scheme

Based on the properties of the exact solution $u(x)$ we construct an *apriori* mesh $\Omega^N \equiv \{0 = x_0 < x_1 < \cdots < x_N = 1\}$ with the mesh points

$$x_i = \begin{cases} \left(\dfrac{i}{N}\right)^r, & 0 \leq i \leq \dfrac{N}{2}, \\ \dfrac{1}{2^r} + \dfrac{2\left(1-\frac{1}{2^r}\right)}{N}\left(i-\dfrac{N}{2}\right), & \dfrac{N}{2} < i \leq N, \end{cases} \tag{2.1}$$

where the discretization parameter N is a positive even integer, $r = \frac{1}{\alpha} > 1$. The mesh points (2.1) are more densely at the region near $x=0$ since the exact solution of the fractional differential equations (1.1) and (1.2) may be singular near $x=0$; away from $x=0$ a uniform mesh is used. Pedas et al. [7,14–16] used the mesh $x_i = \left(\frac{i}{N}\right)^r$ for $0 \leq i \leq N$, which may be coarse when i close to N. Our mesh is a slight modification of that in [7,14–16].

It is well known that the initial value problem (1.1) and (1.2) can be written as the following equivalent Volterra integral equation of the second kind with a weakly singular kernel [3, Lemma 6.2]

$$u(x) = u(0) - \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, u(t)) \mathrm{d}t, \tag{2.2}$$

$$u(0) = \gamma. \tag{2.3}$$

In the following we will discrete this integral equation instead of the differential equations (1.1) and (1.2).

An approximation to the integral can be obtained by a quadrature formula

$$\begin{aligned} \int_0^{x_i} (x_i - t)^{\alpha-1} f(t,u(t))\mathrm{d}t &= \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} (x_i - t)^{\alpha-1} f(t,u(t))\mathrm{d}t \\ &\approx \sum_{k=1}^{i} f(x_k, u_k) \int_{x_{k-1}}^{x_k} (x_i - t)^{\alpha-1}\mathrm{d}t \\ &= \frac{1}{\alpha} \sum_{k=1}^{i} f(x_k, u_k)\left[(x_i - x_{k-1})^{\alpha} - (x_i - x_k)^{\alpha}\right]. \end{aligned}$$

Then, we have the following discretization scheme for the problem (2.2) and (2.3):

$$u_i^N = u_0^N - \frac{1}{\Gamma(\alpha+1)} \sum_{k=1}^{i} f\left(x_k, u_k^N\right)\left[(x_i - x_{k-1})^{\alpha} - (x_i - x_k)^{\alpha}\right], \; 1 \leq i \leq N, \tag{2.4}$$

$$u_0^N = \gamma. \tag{2.5}$$