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A quadrature method for numerical solutions of fractional differential equations

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ABSTRACT

In this article, a numerical method is developed to obtain approximate solutions for a certain class of fractional differential equations. The method reduces the underlying differential equation to system of algebraic equations. An algorithm is presented to compute the coefficient matrix for the resulting algebraic system. Several examples with numerical simulations are provided to illustrate effectiveness of the method.

1. Introduction

Fractional derivatives and integrals are non-local operators since both of these operators involve integration which is a non-local operator. This property makes these operators an efficient tool to describe long term memory effects, asymptotic scaling and hereditary properties of various physical phenomena. Many mathematicians put their interest and efforts in the field of fractional calculus and also developed their own definitions for derivatives and integrals of fractional order. The most famous definitions in the world of fractional calculus are the Riemann Liouville, Caputo, Weyl and Grunwald Letnikov definitions. In this paper, we have attached our attention only to Caputo fractional derivatives.

The increasing number of applications of fractional differential equations motivated many researchers and appreciable work has been done to develop efficient methods for exact and numerical solutions of fractional differential equations. Mathematical models are used to understand the behavior of physical phenomena. The best way to solve these models is by using calculus. These are the analytic methods, because we use analysis to figure out the solution. But this tends to work only for simple models. For more complex models, the solution becomes too complicated, it is often necessary to resort to numerical techniques for solving the problem. Many numerical methods are developed to solve fractional differential equations. These methods include homotype perturbation method [1], Wavelets methods [2–4], Adomian decomposition method [5], extrapolation method [6], predictor–corrector method [7], homotype analysis method [8], fractional linear multi-step method [9], generalized differential transform method [10], variational iteration method [11], finite difference method [12], hybrid function method [13], multiquadric radial basis function [14], m-step Methods[15] and fractional convolution quadrature based on generalized Adams methods [16].

In this paper, we develop a numerical method for solving a class of fractional differential equations

$$D_a^{\alpha} u(x) + AD_a^{\beta} u(x) + Bu(x) = f(x), \quad x \in [a, b], \tag{1.1}$$

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$$u(a) = u_0, \quad u'(a) = u_0, \quad u^{(i)}(a) = 0, \quad \text{for} \quad i > 1, \tag{1.2}$$

where $\alpha, \beta \in \mathbb{R}^+$, $\alpha \geq \beta$, $A, B \in \mathbb{R}$ and $f(x)$ is known functions. We also discuss existence and uniqueness of solutions for the problem (1.1) under some weaker assumptions on function f. Motivated by work in [17] we have adopted the technique of combining quadrature rules. The quadrature rules we have combined are the trapezoidal rule and Simpson rule. The reason behind combining these two rules is Simpson rule is only applicable to an even number of intervals and the trapezoidal rule is less accurate. After combining these two rules we can implement it on any number of intervals and also the error estimate can be minimized.

In the first section we have discussed some basic definitions and properties of fractional order derivatives and integrals. Existence and uniqueness of solutions of the problem is discussed in the second section. The third section involves development of numerical method. The fourth section sticks to convergence of the numerical method and the fifth section is about implementation of numerical method to various examples.

2. Preliminaries

In this section we introduce some necessary definitions and mathematical preliminaries of fractional calculus [18,19].

Definition 2.1. Let $\alpha \in \mathbb{R}^+$. The operator I_a^α, defined on $L_1[a, b]$ by

$$I_a^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} u(t) dt,$$

is called the Reimann–Liouville fractional integral operator of order α.

Definition 2.2. Let $\alpha \in \mathbb{R}^+$, $n = \lceil \alpha \rceil$ and $u \in AC^n[a, b]$. The operator ${}^{RL}D_a^\alpha$, defined by

$${}^{RL}D_a^\alpha u(x) = \frac{d^n}{dx^n} \int_a^x \frac{(x-t)^{n-\alpha-1}}{\Gamma(n-\alpha)} u(t) dt,$$

is called the Reimann–Liouville fractional derivative of order α.

Definition 2.3. Let $\alpha \in \mathbb{R}^+$, $n = \lceil \alpha \rceil$ and $u \in AC^n[a, b]$. Then the Caputo fractional derivative of $u(x)$ is defined by

$$D_a^\alpha u(x) = \int_a^x \frac{(x-t)^{n-\alpha-1}}{\Gamma(n-\alpha)} u^{(n)}(t) dt.$$

Under natural conditions on the function $u(x)$, for $\alpha \to n$ the fractional derivative ${}^{RL}D_a^\alpha$ or D_a^α becomes a conventional nth order derivative of the function $u(x)$. In the following, we state some important properties of fractional operators that will be used in the sequel.

Lemma 2.4. *Let* $\alpha > 0$, $\beta > -1$ *and* $f(x) = (x-a)^\beta$, *then* $I_a^\alpha f(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (x-a)^{\alpha+\beta}$.

Lemma 2.5. *Let* $\alpha, \beta \in \mathbb{R}^+$ *and* $u \in L_1[a, b]$ *then* $I_a^\alpha I_a^\beta u(x) = I_a^{\alpha+\beta} u(x)$ *holds almost everywhere on* $[a, b]$.

Lemma 2.6. *If* $\alpha \in \mathbb{R}^+$ *and* $u \in C[a, b]$, *then* $D_a^\alpha I_a^\alpha u(x) = u(x)$.

Lemma 2.7. *Let* $\alpha > 0$, $n = \lceil \alpha \rceil$ *and* $u \in AC^n[a, b]$ *then* $I_a^\alpha D_a^\alpha u = u(x) - \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} u^{(k)}(a)$.

3. Existence and uniqueness

In this section we investigate the question of existence and uniqueness of solution for problems (1.1) and (1.2). The existence and uniqueness of the solution to linear and nonlinear fractional differential equations have been thoroughly investigated in [18,19]. Particularly, the questions of existence and uniqueness of the solution to initial value problems (1.1) and (1.2) have been discussed in [20]. It is to be noted that in [20] the function $f(x)$ in Eq. (1.1) is assumed to be continuous. We present here existence result for problems (1.1) and (1.2) under weaker assumptions.

Integral equations are extensively used to study the qualitative properties of differential equation. At this point we establish an equivalence result between (1.1), (1.2) and Voltera integral equation.

Theorem 3.1. *Assume* $f \in L_1[a, b]$, *then a function u is solution of initial value problems (1.1) and (1.2) if and only if u is solution of the integral equation*

$$u(x) + \int_a^x K(x, t) u(t) dt = F(x), \tag{3.1}$$

where