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# A unified numerical scheme for the multi-term time fractional diffusion and diffusion-wave equations with variable coefficients

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ABSTRACT

We consider the numerical solutions of the multi-term time fractional diffusion and diffusion-wave equations with variable coefficients in a bounded domain. The time fractional derivatives are described in the Caputo sense. A unified numerical scheme based on finite difference method in time and Legendre spectral method in space is proposed. Detailed error analysis is given for the fully discrete scheme. The convergence rate of the proposed scheme in $L^2$ norm is $O(\tau^2 + N^{1-m})$, where $\tau$, $N$, and $m$ are the time-step size, polynomial degree, and regularity in the space variable of the exact solution, respectively. Numerical examples are presented to illustrate the theoretical results.



## 1. Introduction

Fractional calculus involves investigating the properties and applications of the derivatives and integrals with non-integer orders. One can refer to [1] for an extensive list of recent applications and mathematical developments of the fractional calculus. Fractional differential equations are the equations involving the fractional derivatives of the unknown functions. There are many kinds of definitions for the fractional derivatives, such as Riemann–Liouville derivative, Caputo derivative, Grünwald–Letnikov derivative, etc.

The time fractional diffusion and diffusion-wave equations are the usual diffusion and wave equations with their first-order time derivative and second-order time derivative replaced by fractional derivatives of order $0 < \alpha < 1$, $1 < \alpha < 2$, respectively [2]. Both analytical and numerical investigations of them have been studied by many authors. For the solution theory of the time fractional diffusion and diffusion-wave equations, one can refer to [3–7]. For the numerical approximation of the time fractional diffusion and diffusion-wave equations, see [8–14], etc.

In this paper we consider the following multi-term time fractional diffusion and diffusion-wave equations with variable coefficients:

$$ {}_0^C D_t^{\gamma} u(x,t) + \sum_{i=1}^{s} b_i {}_0^C D_t^{\gamma_i} u(x,t) = \mathcal{L}u(x,t) + g(x,t), \quad -1 < x < 1,\ 0 < t \le T, \tag{1.1} $$

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where

$$\mathcal{L}u = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) - q(x)u,$$

$$p \in C^1[-1,1],\ q \in C[-1,1],\ p(x) > 0,\ q(x) \geq 0,\ x \in [-1,1],$$

$$0 < \gamma_s < \cdots < \gamma_1 < \gamma < 2, \quad b_i \geq 0,\ i = 1,\ldots,s,\ s \in \mathbb{N}_0,$$

and ${}_0^CD_t^\gamma u(x,t)$ is the Caputo fractional derivative of order $\gamma$ with respect to $t$, its exact definition will be given in next section. We endow Eq. (1.1) with the following boundary conditions:

$$u(-1,t) = 0, \qquad u(1,t) = 0, \qquad 0 \leq t \leq T, \tag{1.2}$$

and initial conditions

$$u(x,0) = u_0(x), \qquad x \in (-1,1), \tag{1.3}$$

$$u_t(x,0) = \psi(x), \qquad x \in (-1,1) \ \text{for } 1 < \gamma < 2. \tag{1.4}$$

In this paper, in the case $0 < \gamma_s < \cdots < \gamma_1 < \gamma < 1$, (1.1) is called the multi-term time fractional diffusion equation. When $1 < \gamma_s < \cdots < \gamma_1 < \gamma < 2$, (1.1) is called the multi-term time fractional diffusion-wave equation. When $0 < \gamma_s < \cdots < \gamma_i < 1 < \gamma_{i-1} < \cdots < \gamma_1 < \gamma < 2$, (1.1) is called the multi-term time fractional mixed diffusion and diffusion-wave equation. Luchko [15] considered the initial–boundary value problems for the generalized multi-term time-fractional diffusion equation, and showed some existence and uniqueness results. Jiang et al. [16] derived the analytical solutions for the multi-term time–space Caputo–Riesz fractional advection–diffusion equations on a finite domain. Li et al. [17] presented the well-posedness and the long-time asymptotic behaviour of the initial–boundary value problems for the multi-term time-fractional diffusion equations. Ding and Nieto [18] used Laplace transform and Fourier transform methods to obtain the analytical solutions of the multi-term time–space fractional reaction–diffusion equations on the whole line, and presented the results in a compact and elegant form in terms of Mittag-Leffler functions.

Liu et al. [19] proposed some computationally effective numerical methods for simulating the multi-term time fractional wave-diffusion equations. Jin et al. [20] used a Galerkin finite element method to approximate the multi-term time fractional diffusion equation on a bounded convex polyhedral domain, and analysed the stability and error estimate for the semi-discrete and fully discrete schemes. Ren and Sun [21] presented a compact difference method for the multi-term time fractional diffusion-wave equation on one-dimensional and two-dimensional bounded domains.

However, the temporal accuracy of the previous methods is depending on the order of the fractional derivatives, and is usually less than two. There are also some papers in which second order discretization was proposed for the time discretization of the fractional derivative operators, see [22–25]. But the high order approximations for single fractional operator either cannot be directly applied to multi-term fractional operators, or the error analysis of them is hard to analyse. Most importantly, they are not workable for solving both the time fractional diffusion and diffusion-wave equations. Huang and Yang [26] proposed a unified difference-spectral method for the single term time–space fractional diffusion equations, but its extension to the multi-term cases is not clear. Recently, Tian et al. [27] proposed a class of second order approximations, called weighted and shifted Grünwald difference (WSGD) operators, for the Riemann–Liouville fractional derivatives. In this paper, we propose a unified numerical scheme which has second order accuracy in time and spectral accuracy in space for the problem (1.1)–(1.4) . The proposed scheme is based on finite difference method in the temporal direction and Legendre spectral method in the spatial direction. More precisely, for the multi-term time fractional diffusion and diffusion-wave equations, we first transform them into equivalent forms with the Riemann–Liouville fractional derivative operator and Riemann–Liouville fractional integral operator, respectively. Then we use weighted and shifted Grünwald difference (WSGD) operators to approximate the fractional operators, and based on a Crank–Nicolson technique, the convergence rate of the fully discrete scheme in $L^2$ norm is $O(\tau^2 + N^{1-m})$, where $\tau$, $N$, and $m$ are the time-step size, polynomial degree, and regularity in the space variable of the exact solution, respectively. The stability and convergence of the fully discrete scheme are rigorously established.

The rest of the paper is organized as follows. In Section 2, some preliminaries and notations are shown. In Section 3, we construct a unified numerical scheme for the multi-term time fractional diffusion and diffusion-wave equations. In Section 4, the stability and convergence of the fully discrete scheme are analysed. We do some numerical experiments in Section 5. Finally, the summary and discussion are presented in Section 6.

## 2. Preliminaries and notations

Let $\Lambda = (-1,1)$. Throughout this paper, we use the usual Sobolev spaces $W^{r,p}(\Lambda)$ with norm $\|\cdot\|_{r,p}$. When $p = 2$, we denote $W^{r,2}(\Lambda)$ and its inner product, semi-norm, and norm by $H^r(\Lambda)$, $(\cdot,\cdot)_r$, $|\cdot|_r$, and $\|\cdot\|_r$, respectively. In particular, $(\cdot,\cdot) = (\cdot,\cdot)_0$, $\|\cdot\| = \|\cdot\|_0$. Furthermore, we denote

$$H_0^1(\Lambda) = \left\{ v \in H^1(\Lambda),\ v(\pm 1) = 0 \right\}.$$