Journal of Computational and Applied Mathematics 327 (2018) 306–313

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Existence of positive solutions for the nonlinear elastic beam equation via a mixed monotone operator

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ARTICLE INFO

Article history:
Received 17 February 2017

Keywords:
Elastic beam equation
Positive solution
Mixed monotone operator

ABSTRACT

In this paper, we use a mixed monotone operator method to investigate the existence and uniqueness of positive solution to a nonlinear fourth-order boundary value problem which describes the deflection of an elastic beam with the left extreme fixed and the right extreme is attached to a bearing device given by a known function. Moreover, we present a concrete example illustrating the result obtained.

1. Introduction and preliminaries

Fourth-order two-point boundary value problems have been studied by different authors because they describe the deflection of an elastic beam (see [1–12] and the references therein, for example).

The main tool in our study is a mixed monotone operator method. This technique was used by the authors in [12] in the study of the following nonlinear boundary value problem

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t)), \quad t \in (0, 1), \\ u(0) = 0, \qquad u'(0) = \mu h(u(0)), \\ u''(1) = 0, \qquad u'''(1) = \mu g(u(1)), \end{cases} \tag{1}$$

where $\lambda > 0$, $\mu \geq 0$ and f, g, h are functions given.

In [10], the authors study the existence and uniqueness of positive solutions for the nonlinear boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), \quad t \in (0, 1), \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) = g(u(1)), \end{cases} \tag{2}$$

where f, g are functions given, by using a mixed monotone operator method.

Using the same technique in [11] the authors study the existence and uniqueness of positive solutions for the boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t)), \quad t \in (0, 1), \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) = g(u(1)), \end{cases} \tag{3}$$

where f, g are functions given.

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http://dx.doi.org/10.1016/j.cam.2017.04.031

Motivated by the above mentioned papers, by using a mixed monotone operator method, we study the existence and uniqueness of positive solutions for the following boundary value problem.

$$\begin{cases} u^{(4)}(t) = f(t, u(t), (Hu)(t)), \quad t \in [0, 1], \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) = g(u(1)), \end{cases} \tag{4}$$

where $f : [0, 1] \times [0, \infty) \times [0, \infty) \to [0, \infty)$, $g : [0, \infty) \to (-\infty, 0]$ are continuous functions and H is an operator (not necessarily linear) applying $C[0, 1]$ into itself and satisfying certain assumptions.

Next, we present some definitions, notations and results which will be used in the proof of our main result.

Suppose that $(E, \|\cdot\|)$ is a real Banach space. A nonempty closed convex set $P \subset E$ is said to be a cone if it satisfies:

1. $x \in P$ and $\lambda \geq 0 \Rightarrow \lambda x \in P$.
2. $-x, x \in P \Rightarrow x = \theta_E$,

where θ_E denotes the zero element of E.

Suppose that P is a cone in the Banach space $(E, \|\cdot\|)$. Then P induces a partial order in E given by,

$$x, y \in E, \quad x \leq y \Leftrightarrow y - x \in P.$$

By $x < y$, we denote $x \leq y$ and $x \neq y$.

If interior of P, $\mathring{P}$, is nonempty, we say that the cone P is solid.

If there exists a constant $C > 0$ such that, for any $x, y \in E$ with $\theta_E \leq x \leq y$ implies $\|x\| \leq C\|y\|$ then the cone P is said to be normal. In this case, the smallest constant C satisfying the last inequality is called the normality constant of P.

For $x, y \in E$, we denote $x \sim y$ when there exist constants $\lambda, \mu > 0$ such that

$$\lambda y \leq x \leq \mu y.$$

Clearly $\sim$ is an equivalence relation.

For $\theta_E < h$, we denote by P_h the set

$$P_h = \{x \in E : x \sim h\}.$$

It is easily seen that $P_h \subset P$.

Definition 1. An operator $T : E \longrightarrow E$ is said to be increasing (resp. decreasing) if, for any $x, y \in E$, $x \leq y$ implies $Tx \leq Ty$ (resp. $Tx \geq Ty$).

Definition 2. An operator $A : P \times P \longrightarrow P$ is called mixed monotone if $A(x, y)$ is increasing in x and decreasing in y, i.e., for any $(x, y), (u, v) \in P \times P$,

$$x \leq u \quad \text{and} \quad y \geq v \Rightarrow A(x, y) \leq A(u, v).$$

Definition 3. An operator $B : P \longrightarrow P$ is called subhomogeneous if

$$B(tx) \geq tBx, \quad \text{for any } t \in (0, 1) \text{ and } x \in P.$$

The following result appears in [13] and it gives us the mixed monotone operator method which we will use in the proof of our main result.

Theorem 1. *Suppose that $\alpha \in (0, 1)$, $h \in E$ with $\theta_E < h$, and P is a normal cone in the Banach space $(E, \|\cdot\|)$. Let $A : P \times P \longrightarrow P$ be a mixed monotone operator such that*

$$A(tx, t^{-1}y) \geq t^{\alpha} A(x, y), \quad \text{for any } t \in (0, 1) \text{ and } x, y \in P.$$

Let $B : P \longrightarrow P$ be an increasing subhomogeneous operator.

Assume that

(i) *there exists $h_0 \in P_h$ such that $A(h_0, h_0) \in P_h$ and $Bh_0 \in P_h$,*
(ii) *there exists a constant $\delta_0 > 0$ such that*

$$A(x, y) \geq \delta_0 Bx, \quad \text{for any } x, y \in P$$

then

(a) *$A : P_h \times P_h \longrightarrow P_h$ and $B : P_h \longrightarrow P_h$,*
(b) *there exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that $rv_0 \leq u_0 \leq v_0$ and*

$$u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0,$$