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# A nonlinear splitting algorithm for systems of partial differential equations with self-diffusion

Matthew A. Beauregard [a,*], Joshua Padgett [b], Rana Parshad [c]

[a] Department of Mathematics and Statistics, Stephen F. Austin State University, Nacogdoches, TX, 75962, United States
[b] Department of Mathematics, Baylor University, Waco, TX, 76798, United States
[c] Department of Mathematics, Clarkson University, Potsdam, NY, 13699, United States

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ABSTRACT

Systems of reaction–diffusion equations are commonly used in biological models of food chains. The populations and their complicated interactions present numerous challenges in theory and in numerical approximation. In particular, self-diffusion is a nonlinear term that models overcrowding of a particular species. The nonlinearity complicates attempts to construct efficient and accurate numerical approximations of the underlying systems of equations. In this paper, a new nonlinear splitting algorithm is designed for a partial differential equation that incorporates self-diffusion. We present a general model that incorporates self-diffusion and develop a numerical approximation. The numerical analysis of the approximation provides criteria for stability and convergence. Numerical examples are used to illustrate the theoretical results.



## 1. Introduction

This paper is motivated by a three-species food chain model first developed in [1] and analyzed in [2]. Recently, this model was improved to consider overcrowding effects of the population species in [3]. Our goal of this paper is to develop reliable, accurate, efficient, and valid numerical approximations that incorporate the nonlinear overcrowding term for the top predator.

Consider an invasive species $r$ that has invaded a certain two dimensional habitat. Let $r$ predate on a middle predator $v$, which in turn predates on a prey $u$. A partial differential equation that includes overcrowding is,

$$\partial_t r = d_3 \Delta r + d_4 \Delta r^2 + cr^2 - w_3 \frac{r^2}{v + D_3} \equiv d_3 \Delta r + d_4 \Delta r^2 + h(u, v, r), \tag{1.1}$$

$$\partial_t v = d_2 \Delta v - a_2 v + w_1 \frac{uv}{u + D_1} - w_2 \frac{vr}{v + D_2} \equiv d_2 \Delta v + g(u, v, r), \tag{1.2}$$

$$\partial_t u = d_1 \Delta u + a_1 u - b_2 u^2 - w_0 \frac{uv}{u + D_0} \equiv d_1 \Delta u + f(u, v, r), \tag{1.3}$$

defined on $\mathbb{R}^+ \times \Omega$. Here $\Omega \subset \mathbb{R}^2$ and $\Delta$ is the two dimensional Laplacian operator. We define $\mathbf{x}$ to be the spatial coordinate vector in two dimensions. The parameters $d_1$, $d_2$ and $d_3$ are positive diffusion coefficients. The initial populations are

* Corresponding author.
*E-mail address:* beauregama@sfasu.edu (M.A. Beauregard).

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**Table 1**
List of parameters used in the three species food chain model. All these parameters are positive constants.

| Symbols | Meaning |
|---|---|
| $u$ | Prey |
| $v$ | Middle predator |
| $r$ | Top predator |
| $a_1$ | Growth rate of prey $u$ |
| $a_2$ | Measures the rate at which $v$ dies out when there is no $u$ to prey on and no $r$ |
| $w_i$'s | Maximum value that the per-capita rate can attain |
| $D_0, D_1$ | Measure the level of protection provided by the environment to the prey |
| $b_2$ | Measure of the competition among prey, $u$ |
| $D_2$ | Value of $v$ at which its per capita removal rate becomes $w_2/2$ |
| $D_3$ | Loss in $r$ due to the lack of its favorite food, $v$ |
| $c$ | Growth rate of $r$ via sexual reproduction |
| $d_4$ | The strength of the overcrowding term |

given as

$$u(0,\mathbf{x}) = u_0(\mathbf{x}), \qquad v(0,\mathbf{x}) = v_0(\mathbf{x}), \qquad r(0,\mathbf{x}) = r_0(\mathbf{x}) \quad \mathbf{x} \in \Omega,$$

are assumed to be nonnegative and uniformly bounded on $\Omega$. Appropriate boundary conditions are specified. Here, we examine homogeneous Dirichlet boundary conditions, however our analysis extends to homogeneous Neumann boundary conditions in a straightforward manner. The parameter definitions are given in Table 1.

This model is rich in dynamics and stems from the Leslie–Gower formulation [4], that is, the middle predator is depredated at a Holling type II rate, and the generalist top predator grows logistically as $cr$, and loses due to intraspecies competition as $-w_3 r^2/(v + D_3)$. The literature is abundant with investigations of variants to this model [5–16]. However, the development and analysis of accurate and efficient numerical approximations has not been considered, especially in situations involving the overcrowding term. The overcrowding term can be viewed as a severe penalty to crowding in the top predator forcing a strong movement to lower concentrations of $r$.

While the above model motivates this paper we develop a nonlinear algorithm for

$$u_t = \Delta\left(u + u^2\right) + f(u, \mathbf{x}, t), \tag{1.4}$$

where $\Delta$ is the standard 2-dimensional Laplacian, $f(u, \mathbf{x}, t)$ is a nonlinear, nonnegative reactive term on $W^{1,\infty}(\Omega)$, and appropriate initial and boundary conditions are given for $u(\mathbf{x}, t)$. Lifting the restriction on the type of reaction terms is a subject of our future work. The primary objective of this research is to develop a nonlinear operator splitting scheme that efficiently approximates the diffusion terms.

This paper is organized as follows. In Section 2 we present the nonlinear variable time splitting model which is based on a modified Douglass–Gunn splitting method. Section 3 details our numerical analysis of the proposed algorithm. It is shown that the method is stable and second-order convergent in time under reasonable criteria for the temporal and spatial sizes. Section 4 contains examples that illustrate our theoretical results and explores the dynamics of (1.1)–(1.3), in particular the effect of the self-diffusion and overcrowding on the numerical solution. Section 5 summarizes our key results.

In the ensuing discussion all lowercase bold letters indicate vectors, uppercase letters are used for matrices. The $\ell^2$-norm is used throughout discussions unless otherwise specified. That is, given a vector $\mathbf{x} \in \mathbb{R}^n$, then

$$\|\mathbf{x}\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}.$$

The matrix norms considered will be the spectral norm, which is induced by the above vector norm.

In the following discussions, we define a scheme as *computationally efficient* if it is *second order accurate* in space and time or better and the *number of operations per time step is directly proportional to the number of unknowns.*

## 2. Nonlinear model

We consider the following model

$$u_t = \Delta\left(u + u^2\right) + f(u, \mathbf{x}, t), \tag{2.1}$$

where $\Delta$ is, in this case, the 2-dimensional Laplacian, $\mathbf{x} = (x, y)$, and appropriate initial conditions are given. Homogeneous Dirichlet boundary conditions are assumed. Without loss of generality, we assume a square domain $\Omega = (0, 1) \times (0, 1)$ in the following discussions.

Given $N \gg 0$, we may inscribe over $\Omega$ the mesh $\mathcal{D}_h = \{(x_i, y_j) \mid i, j = 0, 1, \ldots, N+1\}$, where $h = 1/(N+1)$ and $x_i = ih$ and $y_i = jh$ for $i, j = 0, 1, \ldots, N+1$. Further, we define $u_{i,j}(t)$ as the approximation to the exact solution $u(x_i, y_j, t)$ and let $\mathbf{v} = (u_{1,1}(t), u_{2,1}(t), \ldots, u_{N,1}(t), \ldots, u_{N,N}(t))^\top$. Similarly, let $\mathbf{f} = (f(u(x_1, y_1, t),$