Journal of Computational and Applied Mathematics 326 (2017) 1–14

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On supracontergence phenomenon for second order centered finite differences on non-uniform grids

Gayaz Khakimzyanov [a], Denys Dutykh [b,*]

[a] Institute of Computational Technologies, Siberian Branch of the Russian Academy of Sciences, Novosibirsk 630090, Russia
[b] LAMA, UMR 5127 CNRS, Université Savoie Mont Blanc, Campus Scientifique, F-73376 Le Bourget-du-Lac Cedex, France

HIGHLIGHTS

- Construction of adaptive grids for BVPs is deeply analyzed.
- It is shown that local mesh refinement does not necessarily lead to low numerical error.
- The stability and convergence of the scheme on non-uniform grids are proved.
- We demonstrate how a 2nd order finite difference scheme can become of the fourth order.

ARTICLE INFO

Article history:
Received 8 November 2016
Received in revised form 9 April 2017

MSC:
primary 65N06
secondary 76M20
65L10

Keywords:
Finite differences
Non-uniform grids
Boundary layer
Boundary value problems
Supraconvergence

ABSTRACT

In the present study we consider an example of a boundary value problem for a simple second order ordinary differential equation, which may exhibit a boundary layer phenomenon depending on the value of a free parameter. To this equation we apply an adaptive numerical method on redistributed grids. We show that usual central finite differences, which are second order accurate on a uniform grid, can be substantially upgraded to the fourth order by a suitable choice of the underlying non-uniform grid. Moreover, we show also that some other choices of the nodes distributions lead to substantial degradation of the accuracy. This example is quite pedagogical and we use it only for illustrative purposes. It may serve as a guidance for more complex problems.

1. Introduction

Boundary layer phenomena are present in many applications, in particular in Fluid Mechanics and Aerodynamics [1]. For instance, the very successful design of the Airbus A320's wing is mainly due to the potential flow theory with appropriate boundary layer corrections [2]. Nowadays this problem is addressed mainly with numerical techniques and it represents serious challenges.

Some numerical approaches to address the boundary layer problem have been proposed since the early 60s. Historically, probably homogeneous schemes on uniform [3] and non-uniform [4] meshes were proposed first by Tikhonov and Samarskii. Later, Il'in introduced the so-called exponential-fitted schemes [5,6], which were generalized recently to finite volumes as well (see *e.g.* [7]). The idea of Il'in consisted in introducing a *fitting factor* into the scheme and requiring that a

* Corresponding author.
E-mail address: Denys.Dutykh@crans.or (D. Dutykh).

http://dx.doi.org/10.1016/j.cam.2017.05.006

particular exact solution satisfies the difference equation exactly. Thanks to pioneering works of Numerov [8,9], we know that on *uniform grids* it is possible to construct fourth order schemes for second order differential equations on three point stencils. However, the application of this scheme to singularly perturbed problems requires the introduction of the so-called *fitting factor*, which may lead to the substantial degradation of the order of convergence. For instance, the uniform (in small parameter) first order convergence was reported in [10, Table 2]. We can also mention two pioneering references where the moving grids were first applied to unsteady problems in shallow water flows [11] and in gas dynamics [12]. The uniform convergence of monotone finite difference operators for singularly perturbed semi-linear equations was shown in [13]. We refer to [14] for a general review of numerical methods in the boundary layer theory.

In [2, pp. 585–586] one can read:

> *I am convinced that it should be possible to develop a general theory of the relation between the grid, the governing equations and the specific solution being computed, but only very hazy ideas how to bring such a theory about.*

Our study is a little attempt towards this research direction. An earlier attempt was undertaken in [15]. Namely, in the present manuscript we consider a singularly perturbed linear second order elliptic ODE as a model equation which exhibits the boundary layer phenomenon. In accordance with the I.M. Gelfand principle, we took the simplest non-trivial example to illustrate our point. The goal is to propose a numerical method for such problems, which is able to solve approximately this problem with an accuracy independent of the value of the perturbation parameter [6]. In the beginning we explain why the classical central finite differences on a *uniform mesh* is not working in practice, even if this method is fully justified from the theoretical point of view with well-known stability and convergence properties [16]. Then, we propose a non-uniform equidistributed grid and we show that the same central difference scheme converges with the fourth order rate on this family of successively refined grids. So, just by changing the distribution of nodes in a smart way one can gain two extra orders of the accuracy! The logarithmically-distributed grids were proposed by Bakhvalov [17]. However, they were shown to converge inevitably with the same second order rate (see [16] for the proof). Later supraconvergence phenomena have been studied theoretically for some elliptic boundary value problems on non-uniform grids [18–20].

Non-uniform grids can be used also to compute numerically blow-up solutions as in the nonlinear Schrödinger equation [21]. See also [22] for a general review of these techniques. There is a related idea of constructing non-uniform grids in order to preserve some or all symmetries of the continuous equation at the discrete level as well [23]. We can only regret that the authors of [23] did not study theoretically the stability and convergence of the scheme depending on symmetry preservation abilities. The same idea holds for invariants [21,24] and asymptotics [25,26]. In the present study we focus essentially on the scheme approximation order depending on the underlying (non-uniform) grid.

The phenomenon of supraconvergence of central finite difference schemes is well known and it was studied rigorously in one spatial dimension in [27] and the 2D case was considered in [28]. The *super-supraconvergence* reported in this manuscript is achieved by using monitoring functions, which depend on lowest order derivatives comparing to examples reported in the literature so far [15,29]. This property greatly simplifies the implementation of grid redistribution methods. As Strang & Iserles [30] discovered the link between the stability and the stencil of a numerical scheme, here we try to understand deeper a link between scheme's convergence order and the underlying grid. In particular, we show that some thoroughly chosen nodes distributions lead to the substantial improvement of the numerical solution accuracy (for a fixed scheme). We show also that some other grid choices (appearing admissible from the first sight) may totally degrade the solution accuracy. These illustrations should serve as an indication for more complex problems.

The present manuscript is organized as follows. The BVP under consideration is described in Section 2. The classical discretization is described in Section 2.1, while the scheme on a general non-uniform grid is provided in Section 2.2. A practical equidistribution method to construct the grids is explained in Section 2.3. A series of numerical experiments on various non-uniform grids is presented in Section 3 and some theoretical insight into these results is given in Sections 4 and 5. Finally, the article is completed by outlining the main conclusions and perspectives of the present study in Section 6.

2. The boundary value problem

Consider the following linear Boundary Value Problem (BVP) for an ordinary differential equation $\mathscr{L}u = 0$ of the second degree with Dirichlet-type boundary conditions on the segment $\mathcal{I} = [\,0,\ \ell\,]$:

$$\mathscr{L}\,u \stackrel{\text{def}}{:=} -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \lambda^2\,u = 0, \qquad u(0) = \mathrm{e}^{-\lambda\ell}, \quad u(\ell) = 1, \tag{2.1}$$

where $\lambda \in \mathbb{R}$ is a model parameter, which can take large (but finite) values.

It can be readily checked that the following function of x solves exactly the BVP (2.1):

$$u(x) = \mathrm{e}^{\lambda(x-\ell)}. \tag{2.2}$$

However, we shall proceed as if the analytical solution (2.2) were not known. It will serve us only to assess the quality of a numerical solution. The peculiarity here is that for sufficiently large values of parameter $\lambda \gg 1$ the solution (2.2) shows a boundary layer type behavior in the vicinity of the point $x = \ell$. It is illustrated in Fig. 1. Similar phenomena occur in Fluid Mechanics where they are of capital importance *e.g.* in Aerodynamics [1]. It justifies the choice of the problem (2.1) in our study.