Journal of Computational and Applied Mathematics 316 (2017) 298–306

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Operator index reduction in electromagnetism

Helia Niroomand Rad

Department of Mathematics, Technical University Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany

ARTICLE INFO

Article history:
Received 15 December 2015
Received in revised form 28 October 2016

Keywords:
Operator differential-algebraic equations
Differentiation index
Galerkin method
Minimal extension technique
Electromagnetism

ABSTRACT

The aim of this work is introducing an index reduction technique in the operator level, thereby regularization of the high index differential-algebraic equations (DAEs) which are derived by spatial semi-discretization of the partial differential equations (PDEs) in electromagnetism can be avoided. The introduced technique is applied to the obtained operator DAE system which is resulted by considering the PDE system in the weak sense for the suitable Hilbert spaces. In addition, for the discretization, the Galerkin method is applied which in turn provides automatically nice properties of the discrete operators for the index determination.

1. Introduction

In this paper, we show that semi-discretization of the partial differential equation (PDE) system of Maxwell equations, in space leads to a high index differential-algebraic equation (DAE) system. In fact, the high index may violate stability and robustness of the time integration, and therefore regularization of such high index DAEs by an index reduction technique before the time integration is necessary. During this work, the differentiation index of the DAE system is briefly called d-index and roughly speaking refers to the minimum needed number of the time derivations we take from the whole equations or from some of them to determine the unknown vector as a continuous vector function of time t and position x, [1, ch. 3].

In the remaining of this paper, the index reduction technique in an operator level applying to the presented operator DAE system of Maxwell equations, obtained by the Maxwell variational formulation, is presented. In fact, by choosing the suitable Hilbert spaces we can write the weak formulation of each PDE system via introducing the corresponding operator DAE system in the Hilbert spaces [2, ch. 8]. In this work, we do not consider the full Maxwell equations but instead investigate only the problematic part of it from DAE index point of view. This part is exploited from the full Maxwell equations and can be solved for the electric field independently from the remaining equations. In fact, the remaining equations automatically leads to a d-index 1 system after semi-discretization in space and therefore does not influence stability of the time integration from DAE index point of view. However, we could consider the full Maxwell equations for the unknown vector of the electric and magnetic fields which may lead to a lower d-index but still not d-index 1. In such case, the same procedure as presented in this paper can be applied to. However, considering the current model is comparatively more beneficial, since the smaller problem is attained.

2. PDE setting

We investigate semi-discretization of the full Maxwell equations in space within the homogeneous, linear and isotropic material, [3, ch. 1], where the permittivity ϵ and the permeability μ as well as the conductivity σ are modeled by positive

E-mail address: niroomandrad.helia@yahoo.com.

http://dx.doi.org/10.1016/j.cam.2016.10.033

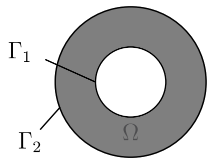

Fig. 1. Schematic view of the computational domain.

constants. Let the computational domain denoted by Ω be bounded, open and connected with the boundary Γ compounded of two boundaries, the inner boundary Γ_1 and the outer boundary Γ_2, see Fig. 1. Additionally, we assume that there exists no impressed current density as well as no electric charge density in Ω. Then, considering the suitable boundary conditions, with sufficiently smooth f and g, for the modeling of electromagnetic coupling, [4], the (full) Maxwell equations [5, ch. 6] read as

$$\begin{aligned} \nabla \cdot E &= 0, && \text{in } \Omega \times \mathbb{I} \\ \nabla \cdot H &= 0, && \text{in } \Omega \times \mathbb{I} \\ \nabla \times E &= -\partial_t \mu H, && \text{in } \Omega \times \mathbb{I} \\ \nabla \times H &= \epsilon \partial_t E + \sigma E, && \text{in } \Omega \times \mathbb{I} \\ \nu \times E &= 0, && \text{on } \Gamma_1 \times \mathbb{I} \\ \nu \times E &= \nu \times f, && \text{on } \Gamma_2 \times \mathbb{I} \\ \nu \cdot H &= g, && \text{on } \Gamma_2 \times \mathbb{I} \\ \nu \cdot H &= 0 && \text{on } \Gamma_1 \times \mathbb{I}. \end{aligned} \tag{1}$$

Therein, $\nu(x)$ denotes the outer normal vector to the domain Ω while $\mathbb{I}$ denotes the bounded observation time interval $[0, T]$ corresponding to the electromagnetic coupling. Instead of considering this system, we investigate two PDE systems achieved by eliminating $H(x, t)$ in the evolution equations and then splitting the obtained PDE system into two PDE systems where each can be evaluated independently for the electric field $E(x, t)$ and the magnetic field $H(x, t)$, i.e.,

$$c\epsilon \partial_t^2 E + \sigma \partial_t E + \mu^{-1} \nabla \times \nabla \times E = 0, \quad \text{in } \Omega \times \mathbb{I} \tag{2a}$$

$$\nabla \cdot E = 0, \quad \text{in } \Omega \times \mathbb{I} \tag{2b}$$

$$\nu \times E = 0, \quad \text{on } \Gamma_1 \times \mathbb{I} \tag{2c}$$

$$\nu \times E = \nu \times f, \quad \text{on } \Gamma_2 \times \mathbb{I} \tag{2d}$$

$$\begin{aligned} \nabla \cdot H &= 0, && \text{in } \Omega \times \mathbb{I} \\ \nu \cdot H &= g, && \text{on } \Gamma_2 \times \mathbb{I} \\ \nu \cdot H &= 0 && \text{on } \Gamma_1 \times \mathbb{I}. \end{aligned}$$

Note that if the initial conditions for E and $\partial_t E$ are given, then the minimum requirements for unique determination of the solution to (2) is fulfilled, since having two sets of boundary conditions, i.e., the set containing (2c) and (2d) as well as the unit set (2b), allow deriving the solution to the second order PDE (2).

From now on, we only work with the problematic set of equations, i.e., (2), which leads to a high index DAE after spatial semi-discretization.

3. Discretization

In this section, we aim at discretization of the PDE system (2) in space by means of Galerkin method. To this end, we introduce the associated weak formulations for that as well as the suitable test spaces and the ansatz space. Then, we look for the Galerkin approximation of the solution $E(x, t)$ in a finite dimensional space included in the ansatz space and show that the Galerkin approximation is satisfying a DAE system of d-index 3 where we mean by d-index the differentiation index, [1, ch. 3].

Let the solution $E(x, t)$ regarding only the first argument x, i.e., $(E(t))(x)$, be in the Hilbert space $\mathcal{V}$, i.e., $E(t) \in \mathcal{V}$. Then with the Lebesgue measure where $\mathbb{I}$ is Lebesgue measurable, e.g., an interval of the real axis, we assume $E \in L^1(\mathbb{I}; \mathcal{V})$ and $E \in L^2(\mathbb{I}; \mathcal{V})$ where for $1 < p \leq \infty$ the Bochner space $L^p(\mathbb{I}; \mathcal{V})$ [6, ch. 5] is defined as

$$L^p(\mathbb{I}; \mathcal{V}) := \left\{ v : \mathbb{I} \mapsto \mathcal{V} | \left(\int_{\mathbb{I}} \|v(t)\|_{\mathcal{V}}^p \, dt \right) < \infty \right\}.$$

The weak derivative of $E \in L^p(\mathbb{I}; \mathcal{V})$ which is denoted by $\partial_t E \in L^p(\mathbb{I}; \mathcal{V})$ is defined such that for all scalar functions $\phi \in C_0^\infty(\mathbb{I})$ the identity

$$\int_{\mathbb{I}} \partial_t \phi(t) E(t) \, dt = - \int_{\mathbb{I}} \phi(t) \partial_t E(t) \, dt,$$