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A plane-wave singularity subtraction technique for the classical Dirichlet and Neumann combined field integral equations

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ABSTRACT

This paper presents expressions for the classical combined field integral equations for the solution of Dirichlet and Neumann exterior Helmholtz problems on the plane, in terms of smooth (continuously differentiable) integrands. These expressions are obtained by means of a singularity subtraction technique based on pointwise plane-wave expansions of the unknown density function. In particular, a novel regularization of the hypersingular operator is obtained, which, unlike regularizations based on Maue's integration-by-parts formula, does not give rise to involved Cauchy principal value integrals. Moreover, the expressions for the combined field integral operators and layer potentials presented in this contribution can be numerically evaluated at target points that are arbitrarily close to the boundary without severely compromising their accuracy. A variety of numerical examples in two spatial dimensions that consider three different Nyström discretizations for smooth domains and domains with corners—one of which is based on direct application of the trapezoidal rule—demonstrates the effectiveness of the proposed higher-order singularity subtraction approach.

1. Introduction

As is well known, boundary integral equation (BIE) methods, such as boundary element methods [5,34] as well as Nyström methods [9,23,24,26,28], provide several advantages over methods based on volume discretization of the computational domain, such as finite difference [36] and finite element methods [20], for the solution of exterior Helmholtz problems. For example, BIE methods can easily handle unbounded domains and radiation conditions at infinity without recourse to approximate absorbing/transparent boundary conditions for truncation of the computational domain [16]. Additionally, BIE methods are based on discretization of the relevant physical boundaries, and they therefore give rise to linear systems of reduced dimensionality—which, although dense, can be efficiently solved by means of accelerated iterative linear algebra solvers [4,9,17,32].

One of the main issues associated with the use of BIE methods is the numerical evaluation of the challenging singular, weakly-singular and nearly-singular integrals that are inherent to the integral operators and layer potentials upon which

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BIE methods are based on. In two spatial dimensions, for example, the single-layer, double-layer and adjoint double-layer operators feature weak $O(\log|\boldsymbol{x}-\boldsymbol{y}|)$ kernel singularities, while the hypersingular operator features a much more stronger $O(|\boldsymbol{x}-\boldsymbol{y}|^{-2})+O(\log|\boldsymbol{x}-\boldsymbol{y}|)$ kernel singularity as $\boldsymbol{y}\to\boldsymbol{x}$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ denote points on the (assumed smooth) boundary. As is known, however, application of the standard regularization procedure, which was originally proposed by Maue [29], enables the hypersingular operator to be expressed in terms of a Cauchy principal value integral that exhibits a $O(|\boldsymbol{x}-\boldsymbol{y}|^{-1})+O(\log|\boldsymbol{x}-\boldsymbol{y}|)$ kernel singularity as $\boldsymbol{y}\to\boldsymbol{x}$. Nearly-singular integrals arise, on the other hand, when integral operators and layer potentials are evaluated at target points close to but not on the boundary of the domain. All these issues greatly hinder the use of BIE methods, as numerical evaluation of integral operators and layer potentials requires special treatment of the kernel singularities by means of specialized quadrature rules and/or semi-analytical techniques for which there is a vast literature that will not be reviewed here; cf. [2,3,5,8,9,13,14,22–24,33,34].

This paper introduces novel expressions for classical uniquely solvable Dirichlet and Neumann Combined Field Integral Equations (CFIEs) [7,11,27,31], that unlike the classical expressions for the CFIEs, are given in terms of operators formulated as integrals of continuously differentiable functions (they in fact exhibit a mild singularity of the form $O(|\boldsymbol{x}-\boldsymbol{y}|^2\log|\boldsymbol{x}-\boldsymbol{y}|)$ as $\boldsymbol{y}\to\boldsymbol{x}$). These expressions are obtained by means of a singularity subtraction technique based on pointwise plane-wave expansions of the unknown density function. In particular, we hereby introduce a regularization of the hypersingular operator that involves neither Cauchy principal value nor weakly singular integrals. The proposed singularity subtraction technique is also utilized to smooth out the nearly-singular integrands that arise from evaluation of layer potentials at target points near the boundary, and from integral operators that result from CFIEs of problems involving two or more obstacles close to each other.

Our singularity subtraction technique relies on the existence of certain homogeneous solutions p_j, $j=0,\dots,N$ $(N>0)$ of the Helmholtz equation, that we refer to as *expansion functions*. Such functions are constructed in such a way that a sufficiently smooth density function, say $\varphi\in C^N(\Gamma)$, can be expressed as $\varphi(\boldsymbol{y})=\sum_{j=0}^N\partial_s^j\varphi(\boldsymbol{x})p_j(\boldsymbol{y}|\boldsymbol{x})+O(|\boldsymbol{x}-\boldsymbol{y}|^{N+1})$ and $\varphi(\boldsymbol{y})=(i\eta)^{-1}\sum_{j=0}^N\partial_s^j\varphi(\boldsymbol{x})\partial_n p_j(\boldsymbol{y}|\boldsymbol{x})+O(|\boldsymbol{x}-\boldsymbol{y}|^{N+1})$ where $\boldsymbol{x}$ is a given point on the boundary Γ and where $\eta>0$ is a constant (the symbols ∂_s and ∂_n denote tangential and normal derivatives on Γ). Calling $K=K_1-i\eta K_2$ the kernel of the Dirichlet or Neumann combined field operators, we can write

$$\int_\Gamma K(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y})=\sum_{j=0}^N\partial_s^j\varphi(\boldsymbol{x})\int_\Gamma\{K_1(\boldsymbol{x},\boldsymbol{y})p_j(\boldsymbol{y}|\boldsymbol{x})-K_2(\boldsymbol{x},\boldsymbol{y})\partial_n p_j(\boldsymbol{y}|\boldsymbol{x})\}\,\mathrm{d}s(\boldsymbol{y})$$
$$+\int_\Gamma\{K_1(\boldsymbol{x},\boldsymbol{y})\rho_1(\boldsymbol{y}|\boldsymbol{x})-K_2(\boldsymbol{x},\boldsymbol{y})\rho_2(\boldsymbol{y}|\boldsymbol{x})\}\,\mathrm{d}s(\boldsymbol{y})$$

where $\rho_1(\boldsymbol{y}|\boldsymbol{x})=\varphi(\boldsymbol{y})-\sum_{j=0}^N\partial_s^j\varphi(\boldsymbol{x})p_j(\boldsymbol{y}|\boldsymbol{x})$ and $\rho_2(\boldsymbol{y}|\boldsymbol{x})=i\eta\varphi(\boldsymbol{y})-\sum_{j=0}^N\partial_s^j\varphi(\boldsymbol{x})\partial_n p_j(\boldsymbol{y}|\boldsymbol{x})$. As it turns out, Green's theorem [12, Theorem 3.1] provides closed-form expressions for the boundary integrals inside the sum. Therefore, the operator $\int_\Gamma K(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y})$ can be easily evaluated by integrating the smoothed mildly singular functions $K_j(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{y}|\boldsymbol{x})$, $j=1,2$—which satisfy $K_j(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{y}|\boldsymbol{x})=O(|\boldsymbol{x}-\boldsymbol{y}|^{N+1}\log|\boldsymbol{x}-\boldsymbol{y}|)$—wherever on Γ the tangential derivatives $\partial_s^j\varphi(\boldsymbol{x})$, $j=1,\dots,N$, exist. In this paper we present a singularity subtraction procedure that considers functions p_j, $j=0,1$, that are obtained explicitly as linear combinations of plane waves.

A singularity subtraction technique similar in principle to the one presented here was originally introduced in [21] for the solution of the Laplace equation and was later extended in [35] to the Helmholtz equation in three spatial dimensions. Both contributions consider direct integral equations derived from Green's representation formula. As such, the integral equations for the Helmholtz equation suffer from spurious resonances in both Dirichlet and Neumann cases (cf. [12, Chapter 3]). The singularity subtraction technique introduced in those references, on the other hand, which provides a smoothing factor that turns weakly singular integrands (in three-dimensions) into bounded but discontinuous functions, does not suffice for the regularization of the hypersingular operator that appears in the combined field integral equation for the Neumann problem.

The structure of this paper is as follows. Section 2 presents the boundary value problems considered in this paper and reviews the definition and main properties of the layer potentials and boundary integral operators. Section 3, subsequently, introduces the smoothed CFIE formulations for both Dirichlet and Nuemann problems. Details on the construction of the expansion functions are provided in Section 4. Finally, Section 5 presents a variety of numerical examples in two spatial dimensions that include three different Nyström discretizations for smooth domains and domains with corners.

2. Preliminaries

This paper considers exterior Helmholtz boundary value problems that arise as an incident TE- or TM-polarized electromagnetic wave impinges on the surface of an axially symmetric perfect electric conductor with cross section $\Omega\subset\mathbb{R}^2$ and boundary $\partial\Omega=\Gamma$. In TE-polarization the scattered field $u_D:\mathbb{R}^2\setminus\Omega\to\mathbb{C}$ is solution of the exterior Dirichlet problem