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Spectral viscosity method with generalized Hermite functions for nonlinear conservation laws

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ABSTRACT

In this paper, we propose new spectral viscosity methods based on the generalized Hermite functions for the solution of nonlinear scalar conservation laws in the whole line. It is shown rigorously that these schemes converge to the unique entropy solution by using compensated compactness arguments, under some conditions. The numerical experiments of the inviscid Burger's equation support our result, and it verifies the reasonableness of the conditions.

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1. Introduction

The spectral methods [9] approximate the exact solution of partial differential equations by seeking an "good" projection in the linear subspace spanned by various orthogonal systems of special functions. The resulting spectral accuracy is highly preferred than any other numerical method, especially when the solution is known to be globally smooth enough. Therefore, they are very appropriate for the elliptic and parabolic equations, thanks to the regularization properties of the operators. When mentioning the nonlinear conservation laws, it is well known that the solution may develop spontaneous jump discontinuity, i.e., shock waves. This irregularity of the solution destroys not only the accuracy of the spectral approximations at the point of discontinuity, but also that in the entire computational domain. It causes the oscillations throughout the domain, which is the so-called Gibb's phenomenon. Moreover, the instability is induced in the nonlinear case. It is shown in [25] that the usual spectral approximate solution may not converge to the entropy solution, the physically relevant one.

Despite all these deficiencies, many mathematicians still pay their efforts to deal with these issues. The problems caused by the irregularity have already been solved for piecewise smooth functions in bounded domain or periodic piecewise smooth function in unbounded domain by filter techniques or reconstruction methods such as the Gegenbauer partial sum, see details in a series of papers [12], [11], [10], [28] and references therein. And the instability of the usual spectral approximations can be avoided by introducing the vanishing viscosity, which was first established by E. Tadmor [24]. The main idea of the spectral viscosity method is the use of artificial diffusion to stabilize the spectral computation without sacrificing its spectral accuracy. The periodic spectral viscosity method has been further investigated in [19], [25] and [20], etc. The nonperiodic Legendre spectral viscosity method is first introduced by Y. Maday et al. [18]. H. Ma proposed the nonperiodic Chebyshev–Legendre spectral viscosity method in [16], [17]. For more literatures related to the spectral viscosity methods with various orthogonal basis in bounded domain, we refer the readers to [6], [8], [13] and references therein.

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As we know, a large amount of physical problems are modeled in unbounded domain. During the past two decades, more attentions were attracted to the numerical solutions of differential equations in unbounded domains. Among the existing literature, the Hermite and Laguerre spectral methods are the most commonly used approaches based on orthogonal polynomials in infinite interval, referring to [7], [29]. Although the Hermite polynomials appear to be a natural choice of orthogonal basis of $L^2(\mathbb{R})$, it is not as popular as Fourier series and Chebyshev polynomials, due to its poor resolution (see [9]) and the lack of the analogue of fast Fourier transformation (FFT), see [4]. However, it is shown in [2] that the poor resolution can be remedied by a suitable choice of scaling factor. Some further investigations on the scaling factor can be found in [26] and Chapter 7, [21]. Recently, a practical guideline of choosing the suitable scaling factors for Gaussian/super-Gaussian functions is summarized by S.S.-T. Yau and the author in [14], where the Hermite spectral method is used to resolve the posterior conditional density function of the states in nonlinear filtering problems.

The literatures on the spectral method in unbounded domains have already been not as rich as those in bounded domains, let alone the spectral viscosity method in unbounded domains. As far as we know, J. Aguirre and J. Rivas [1] is the only paper that considered the spectral viscosity method based on the Hermite functions. However, they defined the Hermite functions in the weighted $L^2_w(\mathbb{R})$, where $w(x) = e^{x^2}$. No scaling factor is introduced there. This essentially causes their involved theoretical proof of the convergence rate of their proposed scheme. Also it is more costly when they try to implement their scheme numerically.

In this paper, we shall revisit the nonlinear scalar conservation laws in $\mathbb{R}$:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = 0, \quad x \in \mathbb{R},\ t > 0 \\ \qquad u(x,0) = u_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $f \in C^1$ is a smooth nonlinear function and $u_0 \in L^\infty(\mathbb{R})$. In general, the spontaneous jump discontinuity may be developed. Therefore, we can not expect the classical solutions to this problem. Moreover, we restrict ourselves to the physically relevant weak solution, the entropy solution, by imposing the entropy condition

$$\frac{\partial U(u)}{\partial t} + \frac{\partial F(u)}{\partial x} \le 0 \tag{1.2}$$

in the sense of distributions, for all entropy pairs (U, F), with $U \in C^2$ convex and $F'(u) = U'(u) f'(u)$, see [23].

We propose Hermite spectral viscosity methods based on generalized Hermite functions with two different viscosity terms.

(I) With viscosity term $\epsilon \partial_x \mathcal{D}_x u$: The approximate solution u_N is obtained by solving

$$\begin{cases} \partial_t u_N + \partial_x (P_{N+1} f(u_N)) - \epsilon_N \partial_x \mathcal{D}_x Q_{m_N} u_N = 0, \quad x \in \mathbb{R},\ t \in (0,T), \\ \qquad\qquad u_N(x,0) = P_N u_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.3}$$

where P_{N+1} is an L^2-orthogonal projection operator defined in (2.15), $\epsilon_N \to 0$ is a positive parameter as N tends to ∞, and Q_{m_N} is a viscosity operator which modifies only the high modes of the Fourier–Hermite expansion. That is,

$$Q_{m_N}\left(\sum_{k=0}^{N} \hat{\phi}_k(t) H_k^\alpha(x)\right) = \sum_{k=0}^{N} \hat{q}_k \hat{\phi}_k(t) H_k^\alpha(x), \tag{1.4}$$

with

$$\begin{cases} \hat{q}_k = 0, \quad \text{if} \quad k \le m_N \\ 1 - \dfrac{m_N}{k} \le \hat{q}_k < 1, \quad \text{if} \quad k > m_N \end{cases}, \tag{1.5}$$

and $m_N < N$ is a positive integer tending to ∞ as N tends to ∞, where $H_k^\alpha(x)$ are the generalized Hermite functions defined in (2.2).

(II) With viscosity term $\epsilon \mathcal{L}_\alpha u$: The approximate solution v_N is obtained by solving

$$\begin{cases} \partial_t v_N + \partial_x (P_{N+1} f(v_N)) + \epsilon_N \mathcal{L}_\alpha v_N = 0, \quad x \in \mathbb{R},\ t \in (0,T), \\ \qquad\qquad v_N(x,0) = P_N u_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.6}$$

where P_{N+1}, ϵ_N are the same as those in (I), and $\mathcal{L}_\alpha$ is defined in (2.4).

Nevertheless, compared to the scheme in [1], the schemes in our paper have at least two advantages:

- Our scheme (II) is numerically stable, due to its symmetry and positivity, while the stability of the scheme in [1] and our scheme (I) can not be guaranteed;

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