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A novel variant of a product integration method and its relation to discrete fractional calculus

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ABSTRACT

A new integration technique, which is suitable for integrands with multiple weak singularities, is introduced. Local truncation errors are given. This scheme, when applied to the Beta function, is shown to emerge naturally from discrete fractional integration. To illustrate the effectiveness of the integration method a numerical example is provided, with somewhat unexpected convergence results.

1. Introduction

Product integration techniques have long been employed for integrating an integrand with an integrable singularity (see e.g. Linz [9], Weiss and Anderssen [11], Cameron and McKee [1], Chandler and Graham [2], Jumarhon and McKee [8] and Diogo et al. [4]). They are used to develop variable coefficient schemes for singular integral equations. For instance, Eggermont [5] writes down the product trapezoidal rule for Abel's integral equation

$$\int_0^t \frac{k(t,s)}{(t-s)^\alpha} y(s)ds = \gamma(t), \quad \alpha \in [0,1), \tag{1}$$

as

$$n \sum_{j=1}^{i} \left(a_{ij}^{(1)} h_{ij} + a_{ij}^{(2)} h_{ij-1} \right) = \gamma(t_i),$$

where

$$a_{ij}^{(1)} = n \int_{t_{j-1}}^{t_j} \frac{s - t_{j-1}}{(t_i - s)^\alpha} ds = \frac{1}{n^{1-\alpha}} \int_0^1 \frac{z}{(i - j + 1 - z)^\alpha} dz,$$

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$$a_{ij}^{(2)} = n \int_{t_{j-1}}^{t_j} \frac{t_j - s}{(t_i - s)^{\alpha}} ds = \frac{1}{n^{1-\alpha}} \int_0^1 \frac{1-z}{(i-j+1-z)^{\alpha}} dz.$$

Here $h_{ij} = k_{ij} y_j$, $y_j = y(j/n)$ and $k_{ij} = k(i/n, j/n)$ are defined on the grid $t_i = i/n$ where $i = 0, 1, \ldots, n$. In this elegant paper he provides a proof of convergence of y_i to $y(x_i)$ as $n \to \infty$ such that i/n remains fixed.

The purpose of this note is to describe what we believe to be a novel type of product integration method which is appropriate for integrands with multiple singularities. The general truncation error is determined. In particular when we apply the technique to the Beta function we shall see that this particular product integration method emerges naturally from the discrete fractional calculus. Indeed it was this observation that suggested this novel product integration method in the first place. Numerical evidence suggests that the order of convergence of this technique is dependent on the particular integral being integrated.

2. A particular product integration method

Consider the integral

$$I(t) = \int_0^t F(s) ds \tag{2}$$

and suppose we can write $F(s) = f(s)g(s)$ where $f(s)$ and $g(s)$ may both have an identifiable finite number of weak (i.e. integrable) singularities.

Let the point t_i belong to the uniform mesh

$$\{t_i = ih : \ i = 0, 1, \ldots, n; \ t_n = 1\}$$

so that (2), when $t = t_n$, may be written as

$$I(t_n) = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} f(s)g(s) ds. \tag{3}$$

Product integration in its simplest form involves assuming that $f(s)$ (or $g(s)$) is constant over the range $[t_{j-1}, t_j]$ and integrating $g(s)$ (or $f(s)$) analytically to obtain a variable coefficient integration rule.

Let us assume instead that $g(t)$, say, has a weak singularity in $[t_{j-1}, t_j]$ and that $g(t)$, $t \in [t_{j-1}, t_j]$, may be approximated by

$$\frac{1}{(t_j - t_{j-1})} \int_{t_{j-1}}^{t_j} g(s) ds. \tag{4}$$

Thus, the integral

$$\int_{t_{j-1}}^{t_j} f(s)g(s) ds$$

can be written as

$$\int_{t_{j-1}}^{t_j} \left(\frac{1}{h} \int_{t_{j-1}}^{t_j} g(u) du \right) f(s) ds = \left(\frac{1}{h} \int_{t_{j-1}}^{t_j} g(u) du \right) \left(\int_{t_{j-1}}^{t_j} f(s) ds \right). \tag{5}$$

Note that the approximation (4) may be applied to any, or all intervals containing (or indeed not containing) an integrable singularity. It will be convenient to give this particular type of product integration a name: we shall call it *product star integration*.

An example of this type of integral is the following:

$$\int_0^{t_n} \ln(s) \ln(t_n - s) ds = t_n (\ln(t_n))^2 - 2t_n \ln(t_n) + (2 - \frac{\pi^2}{6}) t_n.$$

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