Discrete Mathematics 340 (2017) 2307–2317

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The maximum product of sizes of cross-intersecting families

Peter Borg

Department of Mathematics, University of Malta, Malta

ARTICLE INFO

Article history:
Received 14 May 2016
Received in revised form 30 December 2016
Accepted 24 April 2017
Available online 25 May 2017

Keywords:
Intersecting family
Cross-intersecting families

ABSTRACT

A set of sets is called a *family*. Two families $\mathcal{A}$ and $\mathcal{B}$ are said to be *cross-t-intersecting* if each set in $\mathcal{A}$ intersects each set in $\mathcal{B}$ in at least t elements. For a family $\mathcal{F}$, let $l(\mathcal{F}, t)$ denote the size of a largest subfamily of $\mathcal{F}$ whose sets have at least t common elements. We call $\mathcal{F}$ a *($\leq r$)-family* if each set in $\mathcal{F}$ has at most r elements. We show that for any positive integers r, s and t, there exists an integer $c(r, s, t)$ such that the following holds. If $\mathcal{A}$ is a subfamily of a ($\leq r$)-family $\mathcal{F}$ with $l(\mathcal{F}, t) \geq c(r, s, t)l(\mathcal{F}, t+1)$, $\mathcal{B}$ is a subfamily of a ($\leq s$)-family $\mathcal{G}$ with $l(\mathcal{G}, t) \geq c(r, s, t)l(\mathcal{G}, t+1)$, and $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq l(\mathcal{F}, t)l(\mathcal{G}, t)$. We give $c(r, s, t)$ explicitly. Some known results follow from this, and we identify several natural classes of families for which the bound is attained.

© 2017 Elsevier B.V. All rights reserved.

1. Introduction

Unless otherwise stated, we shall use small letters such as x to denote non-negative integers or set elements or functions, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (that is, sets whose members are sets themselves). The set $\{1, 2, \ldots\}$ of positive integers is denoted by $\mathbb{N}$. For any $m, n \in \mathbb{N}$ with $m \leq n$, the set $\{i \in \mathbb{N} : m \leq i \leq n\}$ is denoted by $[m, n]$. We abbreviate $[1, n]$ to $[n]$. We call a set A an *r-element set*, or simply an *r-set*, if its size $|A|$ is r. For a set X, the *power set of* X (that is, the family of subsets of X) is denoted by 2^X, and the family of r-element subsets of X is denoted by $\binom{X}{r}$. It is to be assumed that arbitrary sets and families are *finite*.

We say that a set A *t-intersects* a set B if A and B have at least t common elements. A family $\mathcal{A}$ of sets is said to be *t-intersecting* if every two sets in $\mathcal{A}$ t-intersect. A 1-intersecting family is also simply called an *intersecting family*.

For a family $\mathcal{F}$ and a set T, we denote the family $\{F \in \mathcal{F} : T \subseteq F\}$ by $\mathcal{F}(T)$. We call $\mathcal{F}(T)$ a *t-star of* $\mathcal{F}$ if $|T| = t$. A t-star of a family is the simplest example of a t-intersecting subfamily. We denote the size of a largest t-star of $\mathcal{F}$ by $l(\mathcal{F}, t)$. We denote the set of largest t-stars of $\mathcal{F}$ by $\mathrm{L}(\mathcal{F}, t)$. We say that $\mathcal{F}$ has the *t-star property* if at least one t-star of $\mathcal{F}$ is a largest t-intersecting subfamily of $\mathcal{F}$.

One of the most popular endeavours in extremal set theory is that of determining the size or the structure of a largest t-intersecting subfamily of a given family $\mathcal{F}$. This originated in [26], which features the classical Erdős–Ko–Rado (EKR) Theorem. The EKR Theorem says that for $1 \leq t \leq r$ there exists an integer $n_0(r, t)$ such that for $n \geq n_0(r, t)$, the size of a largest t-intersecting subfamily of $\binom{[n]}{r}$ is $\binom{n-t}{r-t}$, meaning that $\binom{[n]}{r}$ has the t-star property. It also says that the smallest possible $n_0(r, 1)$ is $2r$; among the various proofs of this fact (see [22,26,32,38,42,44]) there is a short one by Katona [44], introducing the elegant cycle method, and another one by Daykin [22], using the Kruskal–Katona Theorem [43,46]. Note that $\binom{[n]}{r}$ itself is intersecting if $n < 2r$. The EKR Theorem inspired a sequence of results [1,28,31,59] that culminated in the complete solution of the problem for t-intersecting subfamilies of $\binom{[n]}{r}$. The solution had been conjectured by Frankl [28]. It particularly tells us that the smallest possible $n_0(r, t)$ is $(t+1)(r-t+1)$; this was established by Frankl [28] for $t \geq 15$, and subsequently by Wilson [59] for any t. Ahlswede and Khachatrian [1] settled the case $n < (t+1)(r-t+1)$. The t-intersection

E-mail address: peter.borg@um.edu.mt.

http://dx.doi.org/10.1016/j.disc.2017.04.019
0012-365X/© 2017 Elsevier B.V. All rights reserved.

problem for $2^{[n]}$ was solved by Katona [42]. These are among the most prominent results in extremal set theory. The EKR Theorem inspired a wealth of results that establish how large a system of sets can be under certain intersection conditions; see [11,24,29,30,35,40,41].

Two families $\mathcal{A}$ and $\mathcal{B}$ are said to be *cross-t-intersecting* if each set in $\mathcal{A}$ t-intersects each set in $\mathcal{B}$. More generally, k families $\mathcal{A}_1, \dots, \mathcal{A}_k$ are said to be *cross-t-intersecting* if for every $i, j \in [k]$ with $i \neq j$, each set in $\mathcal{A}_i$ t-intersects each set in $\mathcal{A}_j$. Cross-1-intersecting families are also simply called *cross-intersecting families*.

For t-intersecting subfamilies of a given family $\mathcal{F}$, the natural question to ask is how large they can be. For cross-t-intersecting families, two natural parameters arise: the sum and the product of sizes of the cross-t-intersecting families (note that the product of sizes of k families $\mathcal{A}_1, \dots, \mathcal{A}_k$ is the number of k-tuples $(A_1, \dots, A_k)$ such that $A_i \in \mathcal{A}_i$ for each $i \in [k]$). It is therefore natural to consider the problem of maximizing the sum or the product of sizes of k cross-t-intersecting subfamilies (not necessarily distinct or non-empty) of a given family $\mathcal{F}$. The paper [14] analyses this problem in general, particularly reducing it to the problem of maximizing the size of a t-intersecting subfamily of $\mathcal{F}$ for k sufficiently large. Solutions have been obtained for various families (see [14]).

Wang and Zhang [58] solved the maximum sum problem for an important class of families that particularly includes $\binom{[n]}{r}$, using a striking combination of the method in [8–10,12,19] and the *no-homomorphism lemma* ([3, Theorem 2], [20, Theorem 3]). The solution for $\mathcal{F} = \binom{[n]}{r}$ and $t = 1$ had been obtained by Hilton [37] and is the first result that addressed the cross-intersection problem described above. The maximum sum problem for $2^{[n]}$ was solved by the author [14, Theorems 3.10, 4.1] via the results in [42,45,58]. The maximum product problem for $2^{[n]}$ was settled by Matsumoto and Tokushige [51] for $k = 2$; the solution extends for the case where $k > 2$ and $n + t$ is even (see [14, Section 5.2], which features a conjecture for the case where $k > 2$ and $n + t$ is odd).

In this paper, we address the maximum product problem for the more general setting where, for each $i \in [k]$, $\mathcal{A}_i$ is a subfamily of a family $\mathcal{F}_i$. This has been considered for a few special families [13,16,39,50,55]. As we explain in Section 2, in many cases it is enough to solve the problem for $k = 2$ (see Lemma 2.5).

The maximum product problem for $\binom{[n]}{r}$ was first addressed by Pyber [55], who proved that for $r, s, n \in \mathbb{N}$ such that either $r = s \leq n/2$ or $r < s$ and $n \geq 2s + r - 2$, if $\mathcal{A} \subseteq \binom{[n]}{r}$ and $\mathcal{B} \subseteq \binom{[n]}{s}$ such that $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{r-1}\binom{n-1}{s-1}$. Subsequently, Matsumoto and Tokushige [50] proved this for $r \leq s \leq n/2$ (see also [5]). For cross-t-intersecting subfamilies, we have the following.

Theorem 1.1 (*[13]*). *For $1 \leq t \leq r \leq s$, there exists an integer $n_0(r, s, t)$ such that for $n \geq n_0(r, s, t)$, if $\mathcal{A} \subseteq \binom{[n]}{r}$, $\mathcal{B} \subseteq \binom{[n]}{s}$, and $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq \binom{n-t}{r-t}\binom{n-t}{s-t}$, and equality holds if and only if $\mathcal{A} = \{A \in \binom{[n]}{r} : T \subseteq A\}$ and $\mathcal{B} = \{B \in \binom{[n]}{s} : T \subseteq B\}$ for some $T \in \binom{[n]}{t}$.*

Hirschorn made a Frankl-type conjecture [39, Conjecture 4] for any r, s, t and n. In [18], a value of $n_0(r, s, t)$ that is close to best possible is established. The special case $r = s$ is treated in [33,56,57], which establish values of $n_0(r, r, t)$ that are also nearly optimal.

Let $c : \mathbb{N}^3 \to \mathbb{N}$ such that for $r, s, t \in \mathbb{N}$, $c(r, s, t) = \max\left\{r\binom{s}{t}, s\binom{r}{t}\right\} + 1$ if $t \leq \min\{r, s\}$, and $c(r, s, t) = 1$ otherwise. Clearly, $c(r, s, t) = r\binom{s}{t} + 1$ for $t \leq r \leq s$.

We call a family $\mathcal{F}$ a $(\leq r)$*-family* if each set in $\mathcal{F}$ has at most r elements. The following is our main result.

Theorem 1.2. *If $r, s, t \in \mathbb{N}$, $\mathcal{F}$ is a $(\leq r)$-family with $l(\mathcal{F}, t) \geq c(r, s, t) l(\mathcal{F}, t + 1)$, $\mathcal{G}$ is a $(\leq s)$-family with $l(\mathcal{G}, t) \geq c(r, s, t) l(\mathcal{G}, t + 1)$, and $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting families such that $\mathcal{A} \subseteq \mathcal{F}$ and $\mathcal{B} \subseteq \mathcal{G}$, then*

$$|\mathcal{A}||\mathcal{B}| \leq l(\mathcal{F}, t) l(\mathcal{G}, t),$$

and if $l(\mathcal{F}, t) l(\mathcal{G}, t) > 0$, then equality holds if and only if $\mathcal{A} = \mathcal{F}(T) \in \mathrm{L}(\mathcal{F}, t)$ and $\mathcal{B} = \mathcal{G}(T) \in \mathrm{L}(\mathcal{G}, t)$ for some t-set T.

Theorem 1.2 is proved in Section 3. As we show in Section 4, it solves the problem for many natural families with a sufficiently large parameter depending on r, s and t. For example, Theorem 1.1 follows from Theorem 1.2 by taking n large enough so that $\binom{n-t}{r-t} \geq c(r, s, t)\binom{n-t-1}{r-t-1}$; see Section 4.1.

For $r, s, t \in \mathbb{N}$, let $\chi(r, s, t)$ be the smallest non-negative real number a such that $|\mathcal{A}||\mathcal{B}| \leq l(\mathcal{F}, t) l(\mathcal{G}, t)$ for every $\mathcal{A}, \mathcal{B}, \mathcal{F}$ and $\mathcal{G}$ such that $\mathcal{F}$ is a $(\leq r)$-family with $l(\mathcal{F}, t) \geq a l(\mathcal{F}, t + 1)$, $\mathcal{G}$ is a $(\leq s)$-family with $l(\mathcal{G}, t) \geq a l(\mathcal{G}, t + 1)$, $\mathcal{A} \subseteq \mathcal{F}$, $\mathcal{B} \subseteq \mathcal{G}$, and $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting.

Problem 1.3. What is the value of $\chi(r, s, t)$?

By Theorem 1.2, $\chi(r, s, t) \leq c(r, s, t)$.

In Theorem 1.2, the case $\mathcal{F} = \mathcal{G}$ is of particular importance. First of all, it implies that $\mathcal{F}$ has the t-star property if $l(\mathcal{F}, t) \geq c(r, r, t) l(\mathcal{F}, t + 1)$.

Theorem 1.4. *If $1 \leq t \leq r$ and $\mathcal{A}$ is a t-intersecting subfamily of a $(\leq r)$-family $\mathcal{F}$ with $l(\mathcal{F}, t) \geq c(r, r, t) l(\mathcal{F}, t + 1)$, then*

$$|\mathcal{A}| \leq l(\mathcal{F}, t),$$

and if $l(\mathcal{F}, t) > 0$, then equality holds if and only if $\mathcal{A} \in \mathrm{L}(\mathcal{F}, t)$.

Download English Version:

https://daneshyari.com/en/article/5776769

Download Persian Version:

https://daneshyari.com/article/5776769

Daneshyari.com