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Exponential independence

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ABSTRACT

For a set S of vertices of a graph G, a vertex u in $V(G)\setminus S$, and a vertex v in S, let $\text{dist}_{(G,S)}(u, v)$ be the distance of u and v in the graph $G-(S\setminus\{v\})$. Dankelmann et al. (2009) define S to be an exponential dominating set of G if $w_{(G,S)}(u)\geq 1$ for every vertex u in $V(G)\setminus S$, where $w_{(G,S)}(u)=\sum_{v\in S}\left(\frac{1}{2}\right)^{\text{dist}_{(G,S)}(u,v)-1}$. Inspired by this notion, we define S to be an exponential independent set of G if $w_{(G,S\setminus\{u\})}(u) < 1$ for every vertex u in S, and the exponential independence number $\alpha_e(G)$ of G as the maximum order of an exponential independent set of G.

Similarly as for exponential domination, the non-local nature of exponential independence leads to many interesting effects and challenges. Our results comprise exact values for special graphs as well as tight bounds and the corresponding extremal graphs. Furthermore, we characterize all graphs G for which $\alpha_e(H)$ equals the independence number $\alpha(H)$ for every induced subgraph H of G, and we give an explicit characterization of all trees T with $\alpha_e(T)=\alpha(T)$.

1. Introduction

Independence in graphs is one of the most fundamental and well-studied concepts in graph theory. In the present paper we propose and study a version of independence where the influence of vertices decays exponentially with respect to distance. This new notion is inspired by the *exponential domination number*, which was introduced by Dankelmann et al. [5] and recently studied in [1–4]. Somewhat related parameters are the well-known *(distance) packing numbers* [8–10] and the *influence numbers* [6,7].

We consider finite, simple, and undirected graphs, and use standard terminology. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The order $n(G)$ of G is the number of vertices of G. The distance $\text{dist}_G(u, v)$ between two vertices u and v in a graph G is the minimum number of edges of a path in G between u and v. If no such path exists, then let $\text{dist}_G(u, v)=\infty$. The diameter $\text{diam}(G)$ of G is the maximum distance between vertices of G. A set of pairwise non-adjacent vertices of G is an independent set of G, and the maximum order of an independent set of G is the independence number $\alpha(G)$ of G.

Let S be a set of vertices of G. For two vertices u and v of G, let $\text{dist}_{(G,S)}(u, v)$ be the minimum number of edges of a path P in G between u and v such that S contains exactly one endvertex of P but no internal vertex of P. If no such path exists, then let $\text{dist}_{(G,S)}(u, v)=\infty$. Note that, if u and v are distinct vertices in S, then $\text{dist}_{(G,S)}(u, u)=0$ and $\text{dist}_{(G,S)}(u, v)=\infty$. For a vertex u of G, let

$$w_{(G,S)}(u)=\sum_{v\in S}\left(\frac{1}{2}\right)^{\text{dist}_{(G,S)}(u,v)-1}, \tag{1}$$

where $\left(\frac{1}{2}\right)^{\infty}=0$. Note that $w_{(G,S)}(u)=2$ for $u\in S$.

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Dankelmann et al. [5] define a set S of vertices to be *exponential dominating* if

$w_{(G,S)}(u) \geq 1$ for every vertex u in $V(G) \setminus S$,

and the *exponential domination number* $\gamma_e(G)$ of G as the minimum order of an exponential dominating set. Analogously, we define S to be *exponential independent* if

$w_{(G,S\setminus\{u\})}(u) < 1$ for every vertex u in S,

that is, the accumulated exponentially decaying influence $w_{(G,S\setminus\{u\})}(u)$ of the remaining vertices in $S \setminus \{u\}$ that arrives at any vertex u in S is strictly less than 1. Let the *exponential independence number* $\alpha_e(G)$ of G be the maximum order of an exponential independent set. An (exponential) independent set of maximum order is *maximum*.

Our results comprise exact values for special graphs as well as tight bounds and the corresponding extremal graphs. Furthermore, we characterize all graphs G for which $\alpha_e(H)$ equals the independence number $\alpha(H)$ for every induced subgraph H of G, and we give an explicit characterization of all trees T with $\alpha_e(T) = \alpha(T)$. We conclude with several open problems.

2. Results

We start with some elementary observations concerning exponential independence. Clearly, every exponential independent set is independent, which immediately implies (i) of the following theorem. The quantity $w_{(G,S\setminus\{u\})}(u)$ does not behave monotonously with respect to the removal of vertices from S. Indeed, if G is a star $K_{1,n-1}$ with center v, and $S = V(G)$ for instance, then $w_{(G,S\setminus\{u\})}(u) = 1$ for every endvertex u of G but $w_{(G,S\setminus\{u,v\})}(u) = \frac{n-2}{2}$, which can be smaller or bigger than 1. In view of this observation part (iii) of the following theorem is slightly surprising.

Theorem 1. *Let G be a graph.*

(i) $\alpha_e(G) \leq \alpha(G)$.
(ii) *If H is a subgraph of G and $S \subseteq V(H)$ is an exponential independent set of G, then S is an exponential independent set of H.*
(iii) *A subset of an exponential independent set of G is an exponential independent set of G.*

Proof. (i) follows from the above observation. Since $\text{dist}_{(G,S\setminus\{u\})}(u, v) \leq \text{dist}_{(H,S\setminus\{u\})}(u, v)$ for every two vertices u and v in S, (ii) follows immediately from (1). We proceed to the proof of (iii). Let S be an exponential independent set of G. Let u and v be distinct vertices in S. In order to complete the proof, it suffices to show

$$w_{(G,S\setminus\{u,v\})}(u) \leq w_{(G,S\setminus\{u\})}(u). \tag{2}$$

For

$$\begin{aligned} S_\infty &= \{w \in S \setminus \{u, v\} : \text{dist}_{(G,S\setminus\{u,v\})}(u, w) = \infty\}, \\ S_= &= \{w \in S \setminus \{u, v\} : \text{dist}_{(G,S\setminus\{u,v\})}(u, w) = \text{dist}_{(G,S\setminus\{u\})}(u, w) < \infty\}, \text{ and} \\ S_> &= \{w \in S \setminus \{u, v\} : \text{dist}_{(G,S\setminus\{u,v\})}(u, w) < \text{dist}_{(G,S\setminus\{u\})}(u, w)\}, \end{aligned}$$

we have $S = \{u, v\} \cup S_= \cup S_> \cup S_\infty$. If $S_> = \emptyset$, then (2) follows immediately from (1). Hence, we may assume that $S_> \neq \emptyset$. Let T be a subtree of G rooted in u such that

- $S_= \cup S_>$ is the set of all leaves of T,
- $\text{dist}_T(u, w) = \text{dist}_{(G,S\setminus\{u,v\})}(u, w)$ for every $w \in S_= \cup S_>$, and
- v is not an ancestor within T of any vertex in $S_=$.

Such a tree can easily be extracted from the union of paths P_w for $w \in S_= \cup S_>$, where P_w is a path of length $\text{dist}_{(G,S\setminus\{u,v\})}(u, w)$ between w and u that intersects $S \setminus \{u, v\}$ only in w, and that avoids v if $w \in S_=$. Since $S_> \neq \emptyset$, the vertex v belongs to T, and the set of leaves of T that are descendants of v is exactly $S_>$. The conditions imposed on T easily imply $\text{dist}_T(u, v) = \text{dist}_{(G,S\setminus\{u\})}(u, v)$. Let $T_>$ be the subtree of T rooted in v that contains v and all its descendants within T. Since S is exponential independent, we obtain $w_{(T_>,S_>)}(v) \leq w_{(G,S\setminus\{v\})}(v) < 1$, which implies

$$\begin{aligned} w_{(G,S\setminus\{u,v\})}(u) &= w_{(T,S_=)}(u) + w_{(T,S_>)}(u) \\ &= w_{(T,S_=)}(u) + \left(\frac{1}{2}\right)^{\text{dist}_T(u,v)} w_{(T_>,S_>)}(v) \\ &< w_{(T,S_=)}(u) + \left(\frac{1}{2}\right)^{\text{dist}_{(G,S\setminus\{u\})}(u,v)} \\ &= \sum_{w\in S_=} \left(\frac{1}{2}\right)^{\text{dist}_{(G,S\setminus\{u\})}(u,w)-1} + \left(\frac{1}{2}\right)^{\text{dist}_{(G,S\setminus\{u\})}(u,v)} \end{aligned}$$