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Refined weight of edges in normal plane maps[☆]

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A R T I C L E I N F O

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Dedicated to the memory of Professor Horst Sachs

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A B S T R A C T

The weight $w(e)$ of an edge e in a normal plane map (NPM) is the degree-sum of its end-vertices. An edge $e = uv$ is of type (i, j) if $d(u) \le i$ and $d(v) \le j$. In 1940, Lebesgue proved that every NPM has an edge of one of the types $(3, 11)$, $(4, 7)$, or $(5, 6)$, where 7 and 6 are best possible. In 1955, Kotzig proved that every 3-connected planar graph has an edge e with $w(e) \le 13$, which bound is sharp. Borodin (1989), answering Erdős' question, proved that every NPM has either a $(3, 10)$-edge, or $(4, 7)$-edge, or $(5, 6)$-edge.

A vertex is simplicial if it is completely surrounded by 3-faces. In 2010, Ferencová and Madaras conjectured (in different terms) that every 3-polytope without simplicial 3-vertices has an edge e with $w(e) \le 12$. Recently, we confirmed this conjecture by proving that every NPM has either a simplicial 3-vertex adjacent to a vertex of degree at most 10, or an edge of types $(3, 9)$, $(4, 7)$, or $(5, 6)$.

By a $k^{(\ell)}$-vertex we mean a k-vertex incident with precisely ℓ triangular faces. The purpose of our paper is to prove that every NPM has an edge of one of the following types: $(3^{(3)}, 10)$, $(3^{(2)}, 9)$, $(3^{(1)}, 7)$, $(4^{(4)}, 7)$, $(4^{(3)}, 6)$, $(5^{(5)}, 6)$, or $(5, 5)$, where all bounds are best possible. In particular, this implies that the bounds in $(3, 10)$, $(4, 7)$, and $(5, 6)$ can be attained only at NPMs having a simplicial 3-, 4-, or 5-vertex, respectively.

1. Introduction

A normal plane map (NPM) is a plane pseudograph in which loops and multiple edges are allowed, but the degree of each vertex and face is at least three.

The degree of a vertex v or a face f, that is the number of edges incident with v or f (loops and cut-edges are counted twice) is denoted by $d(v)$ or $d(f)$, respectively. A *k-vertex* is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k^+-vertex v satisfies $d(v) \ge k$, etc.

An edge uv is an *(i, j)-edge* if $d(u) \le i$ and $d(v) \le j$. By a $k^{(\ell)}$-vertex we mean a k-vertex incident with precisely ℓ triangular faces. An edge uv is an *edge of type* $(i^{(\ell)}, j)$ if u is an $i^{(\ell)}$-vertex and $d(v) \le j$.

By $\delta(G)$ and $w(G)$ we denote the minimum vertex degree and the minimum weight of edges of a graph G, respectively. We will drop the argument when it is clear from context.

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Already in 1904, Wernicke [17] proved that every NPM with $\delta = 5$ satisfies $w \leq 11$. In 1940, Lebesgue [15] proved that every NPM has a $(3, 11)$-edge, or $(4, 7)$-edge, or $(5, 6)$-edge, where 7 and 6 are best possible. In 1955, Kotzig [14] proved that every 3-connected planar graph satisfies $w \leq 13$, which bound is sharp.

In 1972, Erdős (see [11]) conjectured that Kotzig's bound $w \leq 13$ holds for all planar graphs with $\delta \geq 3$. Barnette (see [11]) announced to have proved this conjecture, but the proof has never appeared in print. The first published proof of Erdős' conjecture is due to Borodin [1].

More generally, Borodin [3–5] proved that every NPM contains a $(3, 10)$-, or $(4, 7)$-, or $(5, 6)$-edge (as easy corollaries of some stronger structural facts having applications to coloring of plane graphs, see [6]).

Note that $\delta(K_{2,t}) = 2$ and $w(K_{2,t}) = t + 2$, so w is unbounded if $\delta \leq 2$. To obtain more detailed information on the local structure of plane graphs of minimum degree at least 2, various approaches were developed, one of them being based on the following concept. An induced cycle $v_1 \dots v_{2k}$ in a graph is 2-*alternating* (Borodin [2]) if $d(v_1) = d(v_3) = \cdots = d(v_{2k-1}) = 2$. This notion, along with its more sophisticated analogues (t-alternating subgraph, 3-alternator (Borodin, Kostochka, and Woodall [9]), cycle consisting of 3-paths (Borodin and Ivanova [7]), etc.), turns out to be useful for the study of graph coloring, since it sometimes provides crucial reducible configurations in coloring and partition problems (more often, on sparse plane graphs, see Borodin [6]). Its first application was to show that the total chromatic number of planar graphs with maximum degree Δ at least 14 equals $\Delta + 1$ (Borodin [2]). In particular, forbidding 2-alternating 4-cycles implies $w \leq 17$ (Borodin [1]), while forbidding all 2-alternating cycles implies $w \leq 15$ (Borodin [2]), where both bounds are tight.

In some coloring applications, it is important to find a light edge incident with one or two 5^--faces. Nowadays, the maximum weight of edges is known for many interesting classes of plane graphs (further examples and references can be found in [3–6,10,12,13]).

As proved by Steinitz [16], the 3-connected plane graphs are planar representations of the convex 3-dimensional polytopes, called hereafter 3-polytopes.

Further analogues of the above results may be obtained when taking into account the optimization of sizes of faces surrounding endvertices of light edges. A vertex is *simplicial* if it is completely surrounded by 3-faces. In 2010, Ferencová and Madaras conjectured (in different terms) that every 3-polytope without simplicial 3-vertices has an edge e with $w(e) \leq 12$. Recently, Borodin and Ivanova [8] confirmed this conjecture by proving, moreover, that every NPM has either a simplicial 3-vertex adjacent to a vertex of degree at most 10, or an edge of types $(3, 9)$, $(4, 7)$, or $(5, 6)$.

The purpose of our paper is to refine the result in [8] by taking into account the number of 3-faces at 5^--vertices.

Theorem 1. *Every normal plane map has an edge of one of the following types:* $(3^{(3)}, 10)$, $(3^{(2)}, 9)$, $(3^{(1)}, 7)$, $(4^{(4)}, 7)$, $(4^{(3)}, 6)$, $(5^{(5)}, 6)$*, or* $(5, 5)$*, where all bounds are best possible.*

In particular, this implies that the bounds in $(3, 10)$, $(4, 7)$, and $(5, 6)$ can be attained only at NPMs having a simplicial 3-, 4-, or 5-vertex, respectively.

2. Proving Theorem 1

2.1. Constructions confirming the sharpness in Theorem 1

The sharpness of the term $(3^{(3)}, 10)$ in Theorem 1 follows by putting a 3-vertex into each face of the icosahedron. The sharpness of $(3^{(2)}, 9)$ was confirmed in Borodin–Ivanova [8] (see Fig. 1). In Fig. 2, we see a construction that shows the sharpness of $(3^{(1)}, 7)$. To reach the bound 7 in $(4^{(4)}, 7)$, we take the $(3, 4, 4, 4)$-Archimedean solid, in which every vertex is incident with a 3-face and three 4-faces, and put a 4-vertex into each 4-face. The sharpness of $(4^{(3)}, 6)$ is confirmed by Fig. 3. To see that $(5^{(5)}, 6)$ is best possible, it suffices to put a 5-vertex into every face of the dodecahedron. Finally, the tightness of the term $(5, 5)$ follows from the $(3, 3, 3, 3, 4)$-Archimedean solid, in which every vertex is incident with four 3-faces and one 4-face.

2.2. Discharging

Suppose M' is a counterexample to Theorem 1 with the smallest possible number of vertices. Hence M' is connected. By M denote a counterexample to Theorem 1 with the maximum number of edges on the same vertices as M'. In other words, adding any diagonal to a 4^+-face of M must result in an edge claimed in Theorem 1.

Let V, E, and F be the sets of vertices, edges and faces of M, respectively. Euler's formula $|V| - |E| + |F| = 2$ for M may be rewritten as

$$\sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6) = -12. \tag{1}$$

Every vertex v contributes the *charge* $\mu(v) = d(v) - 6$ to (1), so only the charges of 5^--vertices are negative. Every face f contributes the non-negative *charge* $\mu(f) = 2d(f) - 6$ to (1). Using the properties of M as a counterexample, we define a local redistribution of μ's, preserving their sum, such that the *new charge* $\mu'(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the new charges is, by (1), equal to -12.

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