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Toughness, binding number and restricted matching extension in a graph

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ABSTRACT

A connected graph G with at least $2m + 2n + 2$ vertices is said to satisfy the property $E(m, n)$ if G contains a perfect matching and for any two sets of independent edges M and N with $|M| = m$ and $|N| = n$ with $M \cap N = \emptyset$, there is a perfect matching F in G such that $M \subset F$ and $N \cap F = \emptyset$. In particular, if G is $E(m, 0)$, we say that G is m-extendable. One of the authors has proved that every m-tough graph of even order at least $2m + 2$ is m-extendable (Plummer, 1988). Chen (1995) and Robertshaw and Woodall (2002) gave sufficient conditions on binding number for m-extendability. In this paper, we extend these results and give lower bounds on toughness and binding number which guarantee $E(m, n)$.

1. Introduction

Toughness and binding number are two graph parameters dealing with graph vulnerability. This paper deals with the interaction between each of these two concepts and restricted matching extension.

For a graph G, let $\omega(G)$ denote the number of components of G. A graph G is *t-tough* if $|S| \geq t \cdot \omega(G - S)$ for every subset S of the vertex set $V(G)$ with $\omega(G - S) > 1$. The *toughness* of G, denoted by tough(G), is the maximum value of t for which G is t-tough. (Here we define tough$(K_n) = \infty$, for all $n \geq 1$.) Thus for all graphs G which are not complete, tough$(G) = \min\{|S|/\omega(G - S)\}$, where the minimum is taken over all cutsets S of vertices in G. The notion of toughness was first formulated by Chvátal [13] in connection with the topic of Hamilton cycles in graphs. Since that time, well over one hundred papers have been published on toughness. We refer the interested reader to the comprehensive surveys [8,10].

The first result to appear relating toughness and matching extension was the following.

Theorem A ([18]). *Suppose that G is a graph with $p = |V(G)|$ vertices, where p is even. Let n be a positive integer with $p \geq 2n+2$. Then if* tough$(G) > n$*, G is n-extendable. Moreover, this lower bound on* tough(G) *is sharp for all n.*

The binding number was first defined by Woodall [24]. Let $N_G(S)$ denote the set of neighbors of a set S in a graph G. We define the *binding number* to be the minimum, taken over all $S \subset V(G)$ with $S \neq \emptyset$ and $N_G(S) \neq V(G)$, of the ratios $|N(S)|/|S|$. The binding number of a graph G will be denoted by bind(G).

The first theorem involving binding number and matchings of a graph was due to Anderson [6].

Theorem B ([6]). *If G is a graph of even order and bind$(G) \geq 4/3$, then G contains a perfect matching.*

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On the other hand, Chen [12] was then the first to apply the binding number property to matching extension by proving the next result.

Theorem C ([12]). *If* bind$(G) > \max\{m, (7m + 13)/12\}$, *then G is m-extendable.*

Later, Robertshaw and Woodall [23] proved a theorem which is generally stronger than the above result. We will see the details in Section 3.

It should be pointed out at the outset that, although toughness and binding number may seem at first to be quite similar, in reality, they are quite different. For example, the problem of computing the toughness of a graph is known to be NP-hard [9], whereas the problem of determining the binding number is, in fact, polynomial [14].

For more on the relationship between toughness and binding number, see [7] and [10].

A connected graph G with at least $2m + 2n + 2$ vertices is said to satisfy the property $E(m, n)$ if G contains a perfect matching and for any two sets of independent edges M and N with $|M| = m$ and $|N| = n$, such that $M \cap N = \emptyset$, there is a perfect matching F in G such that $M \subset F$ and $N \cap F = \emptyset$. Historically speaking, the property $E(m, n)$ generalizes the older notion of matching extendability. Indeed one can define a graph to be m-extendable if and only if it satisfies the property $E(m, 0)$. There exist several survey articles on matching extension (cf. [19–21]). The $E(m, n)$ property was first introduced in [22] and the body of work which has developed on this subject falls under the rubric *restricted matching extension*. For an overview of the area we refer the reader to [1–5,16,17].

In this paper, we study the effects of toughness and binding number on restricted matching extension. In Section 2, we prove a sharp bound of toughness which guarantees a graph to be $E(m, n)$. In Section 3, we extend a result in [23] and give a bound on binding number which guarantees a graph to be $E(m, n)$. In Section 4, we compare the results given in Sections 2 and 3, and highlight the difference between toughness and binding number in terms of $E(m, n)$.

For graph-theoretic terminology and definitions not explained in this paper, we refer the reader to [11]. We denote the (vertex)-connectivity and the minimum degree of a graph G by $\kappa(G)$ and $\delta(G)$, respectively. We also denote by $c_o(G)$ the number of odd components of G. By the definition of toughness, it is easy to see $\kappa(G) \geq 2 \cdot \text{tough}(G)$.

2. Toughness

In this section, we study the interaction between toughness and the property $E(m, n)$. As we have seen in the introduction, Plummer [18] proved that a graph G of even order at least $2m + 2$ and tough$(G) > m$ is m-extendable. He also proved that this lower bound on tough(G) is sharp for all m. Later, Liu and Yu [15] studied the relationship between the toughness and the existence of a perfect matching in vertex-deleted and edge-deleted subgraphs.

Theorem D ([15]). *Let G be a graph of even order p and let k be a positive integer with $p \geq 2k + 2$.*

(1) *If* tough$(G) > k$, *then a subgraph of G obtained by deleting any $2k$ vertices has a perfect matching, and*
(2) *if* tough$(G) \geq k$, *then a subgraph of G obtained by deleting any $2k - 1$ edges has a perfect matching.*

They also proved that the bounds on toughness stated in the two hypotheses above are best possible.

Clearly, Theorem D (1) contains Theorem A. Moreover, according to Theorem D (2), for nonnegative integers k and p with p even and $p \geq 4k$, a graph G of order p satisfying tough$(G) \geq k$ is $E(0, 2k - 1)$.

In this section, we prove the following result.

Theorem 1. *Let m and n be nonnegative integers, and let G be a graph of even order at least $2m+2n+2$. If $(m, n) \notin \{(0, 0), (0, 1)\}$ and* tough$(G) > m + \frac{1}{2}n$, *then G is $E(m, n)$.*

Note that we only require tough$(G) > n - \frac{1}{2}$ for a graph G to be $E(0, 2n - 1)$ though Theorem D (2) assumes tough$(G) \geq n$ and this bound is sharp for their conclusion. This is because Theorem D concerns the deletion of any set of edges, whether they are independent or not, while we require our matching to avoid only independent edges.

Note that by Tutte's Theorem, every 1-tough graph of even order contains a perfect matching and satisfies $E(0, 0)$. Though this is an easy fact, we will frequently use it in the subsequent arguments.

Proof of Theorem 1. Since $(m, n) \notin \{(0, 0), (0, 1)\}$, we have $m + \frac{1}{2}n \geq 1$. Therefore, the hypotheses tough$(G) > m + \frac{1}{2}n$ and $|V(G)| \equiv 0 \pmod 2$ guarantee the existence of a perfect matching.

Assume G is not $E(m, n)$. Then there exists a pair of sets of independent edges $\{e_1, e_2, \ldots, e_m\}$ and $\{f_1, f_2, \ldots, f_n\}$ such that G does not contain a perfect matching M with $\{e_1, \ldots, e_m\} \subset M$ and $\{f_1, \ldots, f_n\} \cap M = \emptyset$. Let $S_0 = \bigcup_{i=1}^{m} V(e_i)$ and $F = \{f_1 \ldots, f_n\}$. Let $G' = (G - S_0) - F$. Then G' does not contain a perfect matching. Since $|V(G')|$ is even, by Tutte's Theorem and parity, G' contains a set of vertices S with $c_o(G' - S) \geq |S| + 2$. In particular, $\omega(G' - S) \geq |S| + 2$. Let $s = |S|$. Let C_1, $C_2, \ldots, C_t$ be the components of $G' - S$ $(t \geq s + 2)$. Let F' be the set of edges in F that join different components of $G' - S$. Note $F' \subset F$ and hence $|F'| \leq n$. Let $H = G - (S \cup S_0)$. Then $G' - S = H - F'$. Therefore, $\omega(H) \geq \omega(G' - S) - |F'| \geq t - n$.

Claim 1. *H is connected.*

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