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Facially-constrained colorings of plane graphs: A survey

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ABSTRACT

In this survey the following types of colorings of plane graphs are discussed, both in their vertex and edge versions: facially proper coloring, rainbow coloring, antirainbow coloring, loose coloring, polychromatic coloring, ℓ-facial coloring, nonrepetitive coloring, odd coloring, unique-maximum coloring, WORM coloring, ranking coloring and packing coloring.

In the last section of this paper we show that using the language of words these different types of colorings can be formulated in a more general unified setting.

1. Introduction

There is a very rich literature devoted to the study of various coloring problems of plane graphs. There are several very recent papers that study different types of colorings/labelings of plane graphs where constraints on colorings/labelings are given by faces. In the first part of this paper these different colorings are discussed, both in their vertex and edge versions. Some of them look like artificial-solitary islands. In the second part we show that using the language of words some of them can be formulated in a more general unified setting.

All graphs considered in this paper are simple connected plane graphs provided that it is not stated otherwise. We use a standard graph theory terminology according to Bondy and Murty [12]. However, we recall some important notations.

A *plane* graph is a particular drawing of a planar graph in the Euclidean plane. Let G be a connected plane graph with vertex set $V(G)$, edge set $E(G)$ and face set $F(G)$. Faces of G are open 2-cells. The *boundary* of a face f is the boundary in the usual topological sense. It is a collection of all edges and vertices contained in the closure of f that can be organized into a closed walk in G traversing along a simple closed curve lying just inside the face f. This closed walk is unique up to the choice of initial vertex and direction, and is called the *boundary walk* of the face f (see [61], p. 101). As an example see the graph in Fig. 1 where $v_1, v_1v_2, v_2, v_2v_6, v_6, v_6v_7, v_7, v_7v_2, v_2, v_2v_3, v_3, v_3v_4, v_4, v_4v_5, v_5, v_5v_1, v_1$ is a boundary walk of the face f.

The *size* of a face f is the length (i.e. the number of edges) of its boundary walk. Let f be a face of size k having a boundary walk $v_0e_0v_1e_1 \dots v_{k-1}e_{k-1}v_0$ with $v_i \in V(G)$ and $e_i = v_iv_{i+1} \in E(G)$, $i = 0, \dots, k-1$. A *facial path* of f is any path of the form $v_mv_{m+1} \dots v_{n-1}v_n$, indices modulo k. A *k-path* is a path on k vertices. A *facial trail* of f is any trail of the form $e_me_{m+1} \dots e_{n-1}e_n$, indices modulo k. A *k-trail* is a trail on k edges.

Two vertices (two edges or two faces) are *adjacent* if they are connected by an edge (have a common endvertex or their boundaries have a common edge). A vertex and an edge are *incident* if the vertex is an endvertex of the edge. A vertex (or an edge) and a face are *incident* if the vertex (or the edge) lies on the boundary of the face.

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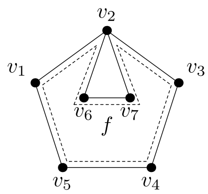

Fig. 1. The boundary walk of the face f.

Two edges of a plane graph G are *facially adjacent* if they are consecutive on the boundary walk of a face of G (for example, the edges v_1v_2 and v_2v_6 in Fig. 1 are facially adjacent but the adjacent edges v_1v_2 and v_2v_7 are not facially adjacent).

Let $\deg(v)$, $g(G)$, $\delta(G)$, $\Delta(G)$ and $\Delta^*(G)$ denote the degree of a vertex v, the girth (i.e. the length of a shortest cycle), the minimum degree, the maximum degree and the maximum face size of G, respectively.

Given two vertices u, v, let $E(uv)$ be the set of edges joining u and v in a multigraph G. The multiplicity $\mu(uv)$ of an edge uv is the size of $E(uv)$. Set $\mu(v) = \max\{\mu(uv) : u \in V(G)\}$, which is called the multiplicity of a vertex v.

An edge (or vertex) *coloring* of G is an assignment of colors to the edges (or vertices) of G, one color to each edge (or vertex). An edge (or vertex) coloring c of a graph G is *proper* if for any two adjacent edges (or vertices) x_1 and x_2 of G, $c(x_1) \neq c(x_2)$ holds. An edge coloring c of a connected plane graph G is *facially proper* if for any two facially adjacent edges e_1 and e_2 of G, $c(e_1) \neq c(e_2)$ holds. Observe that this coloring need not to be proper in a usual sense. We require only that facially adjacent edges must receive different colors. On the other hand these types of colorings coincide in the class of subcubic graphs.

A *list assignment* of a graph G is a function L that assigns a set $L(x)$ of colors to each vertex $x \in V(G)$ (or edge $x \in E(G)$). A vertex (or edge) *L-coloring* is a coloring of G such that each vertex $x \in V(G)$ (or edge $x \in E(G)$) is assigned a color from $L(x)$. A graph G is *k-choosable* (*k-edge-choosable*) if for every list assignment L of G such that $|L(x)| \geq k$ for each vertex $x \in V(G)$ (edge $x \in E(G)$), there is a proper vertex (edge) L-coloring of G. The concept of L-coloring was introduced independently by Vizing [130] and Erdős, Rubin and Taylor [53].

Colorings in this paper need not to be proper.

2. Facially-proper colorings

2.1. Proper vertex-coloring

Historically the first considered coloring problem was the Four Color Problem. It is a problem concerning proper vertex-colorings of plane graphs. The result is the famous Four Color Theorem, which states that the vertices of every plane graph G can be colored with at most four colors so that any two consecutive vertices of any facial walk of G receive distinct colors, see Appel and Haken [5–7] or Robertson et al. [116] for a simpler proof. The bound four is tight as can be seen on the graph of the tetrahedron. For some families of plane graphs this bound can be lowered to three. The first result of this type is the theorem by Grötzsch [62,126] stating that every triangle-free plane graph admits a proper vertex-coloring with at most three colors. Grötzsch's theorem has been a starting point for extended research of plane graphs that are properly colorable with three colors, see e.g. [1,17–20,22,31,63,119,134,137]. One of the open conjectures in this direction is Steinberg's conjecture [123].

Conjecture 1 ([123]). *Every plane graph without 4-cycles and without 5-cycles is 3-colorable.*

For a recent survey concerning this line of the research see a nice survey paper of Borodin [14].

In contrast with Grötzsch's theorem and the Four Color Theorem, Voigt constructed a triangle-free plane graph that is not 3-choosable [131] and a plane graph that is not 4-choosable [132]. On the other hand Thomassen [125] proved that every plane graph is 5-choosable.

2.2. Facially-proper edge-coloring

Facially-proper edge-coloring was first studied for the family of cubic bridgeless plane graphs and for the family of plane triangulations. (Remind that the classical proper edge-coloring and the facially-proper edge-coloring coincide in the class of subcubic graphs.) Already Tait [124] observed that the Four Color Problem is equivalent to the problem of facially-proper 3-edge-coloring of any plane triangulation and to the problem of facially-proper 3-edge-coloring of cubic bridgeless plane graphs (see [117]). It is known that any connected plane graph admits a facially-proper edge-coloring with at most four colors, see [55]. The bound four is tight which can be easily seen on the graph of a wheel on six vertices.

A challenging open problem in this direction is the following.

Problem 1. Characterize all plane graphs that admit a facially-proper edge-coloring with three colors.