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The minimum number of vertices in uniform hypergraphs with given domination number

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ABSTRACT

The *domination number* $\gamma(\mathcal{H})$ of a hypergraph $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ is the minimum size of a subset $D \subset V(\mathcal{H})$ of the vertices such that for every $v \in V(\mathcal{H}) \setminus D$ there exist a vertex $d \in D$ and an edge $H \in \mathcal{E}(\mathcal{H})$ with $v, d \in H$. We address the problem of finding the minimum number $n(k, \gamma)$ of vertices that a k-uniform hypergraph $\mathcal{H}$ can have if $\gamma(\mathcal{H}) \geq \gamma$ and $\mathcal{H}$ does not contain isolated vertices. We prove that

$$n(k, \gamma) = k + \Theta(k^{1-1/\gamma})$$

and also consider the s-wise dominating and the distance-l dominating version of the problem. In particular, we show that the minimum number $n_{dc}(k, \gamma, l)$ of vertices that a connected k-uniform hypergraph with distance-l domination number γ can have is roughly $\frac{k\gamma l}{2}$.

1. Introduction

In this paper we establish basic inequalities involving fundamental hypergraph parameters such as order, edge size, and domination number.

Many problems in extremal combinatorics are of the following form: what is the smallest or largest size that a graph, hypergraph, set system can have, provided it satisfies a prescribed property? In most cases, size is measured by the number of edges, hyperedges, sets, respectively, contained in the object, and the number of vertices is usually included in the prescribed property. However, sometimes it can be interesting and even applicable to consider problems about the minimum or maximum number of vertices [18–20].

In the present paper we address the problem of finding the minimum number of vertices in a k-uniform hypergraph that has a large domination number. The domination number $\gamma(G)$ of a graph G, a widely studied notion (see [10,11]), is the smallest size that a subset $D \subset V(G)$ of the vertices can have if every vertex $v \in V(G) \setminus D$ has a neighbor in D.

We will be interested in the hypergraph version of this notion, which was investigated first in [1] and later studied in [2–4,14,16]. Let $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ be a hypergraph. The *neighborhood*[1] of a vertex $v \in V(\mathcal{H})$ is the set $N_v := \{v\} \cup \bigcup_{E \in \mathcal{E}(\mathcal{H}): v \in E} E$, and the *neighborhood of a set* $S \subset V(\mathcal{H})$ is defined as $N(S) := \bigcup_{v \in S} N_v$. A set $D \subset V(\mathcal{H})$ is called a *dominating set* of $\mathcal{H}$ if $D \cap N_v \neq \emptyset$ for all $v \in V(\mathcal{H})$. Equivalently we can say that D is a dominating set if and only if $N(D) = V(\mathcal{H})$. The

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[1] In this paper we use the short term “neighborhood”, although this is called “closed neighborhood” in the main part of the literature. We note that the inclusion of $\{v\}$ in the definition of N_v may be omitted if v is not an isolated vertex in $\mathcal{H}$.

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minimum size $\gamma(\mathcal{H})$ of a dominating set in a hypergraph $\mathcal{H}$ is *the domination number* of $\mathcal{H}$. As all isolated vertices always are contained in every dominating set, they can be eliminated in an obvious way, therefore we restrict our attention to hypergraphs without isolates.

Let $n(k,\gamma)$ be the minimum number of vertices that a k-uniform hypergraph with no isolated vertices must contain if its domination number is at least γ. Beyond the trivial case of $n(k,1)=k$, the problem of determining $n(k,\gamma)$ is natural and seems to be interesting enough to be addressed on its own right; nevertheless, Gerbner et al. (Problem 17 in [8]) arrived from a combinatorial search-theoretic framework at the particular problem of deciding whether $n(k,3)\geq 2k+3$ holds or not. We answer this problem in the negative, determining the asymptotic behavior of $n(k,\gamma)$ as a function of k for every fixed γ, up to the exact growth order of the second term. To state our result in full strength, we need to introduce two generalizations of domination. For an integer $s>0$ we call $D\subset V(\mathcal{H})$ an *s-dominating set* of $\mathcal{H}$ if $|D\cap N_v|\geq s$ for all $v\in V(\mathcal{H})\setminus D$ and we call D an *s-tuple dominating set* if $|D\cap N_v|\geq s$ for all $v\in V(\mathcal{H})$. Note that dominating sets are exactly the 1-dominating sets and 1-tuple dominating sets. As introduced in [7] and [9], respectively, the minimum size $\gamma(\mathcal{H},s)$ of an s-dominating set in a hypergraph $\mathcal{H}$ is the *s-domination number of* $\mathcal{H}$ and the minimum size $\gamma_\times(\mathcal{H},s)$ of an s-tuple dominating set in a hypergraph $\mathcal{H}$ is the *s-tuple domination number*[2] *of* $\mathcal{H}$. By definition, we have $\gamma(\mathcal{H},s)\leq\gamma_\times(\mathcal{H},s)$. For every pair γ,s of integers with $\gamma\geq s$, let $n(k,\gamma,s)$ denote the minimum number of vertices that a k-uniform hypergraph $\mathcal{H}$ must have if $\gamma(\mathcal{H},s)\geq\gamma$ holds and there exist no isolated vertices in $\mathcal{H}$ and let $n_\times(k,\gamma,s)$ denote the minimum number of vertices that a k-uniform hypergraph $\mathcal{H}$ must have if $\gamma_\times(\mathcal{H},s)\geq\gamma$ holds and there exist no isolated vertices. From the above, we have $n_\times(k,\gamma,s)\leq n(k,\gamma,s)$.

Our main theorem about s-domination is the following.

Theorem 1.1. *For every $\gamma\geq 2$ and $s\geq 1$ with $\gamma>s$ we have*

$$k+k^{1-1/(\gamma-s+1)}\leq n_\times(k,\gamma,s)\leq n(k,\gamma,s)\leq k+(4+o(1))k^{1-1/(\gamma-s+1)}.$$

Another generalization of domination is distance-l domination, which was introduced by Meir and Moon in [17]. This notion has been studied only for graphs so far. A good survey of the results until 1997 is [12]. For more recent upper and lower bounds on the distance-l domination number of graphs see [13] and [6].

In distance-l domination a vertex v dominates all vertices that are at distance at most l from v. As the definition of distance in graphs involves paths, and paths in hypergraphs can be defined in several ways, distance-l domination could be addressed with each of those definitions. But as we will remark in Section 4, only so-called 'Berge paths' offer new problems in our context. A *Berge path* of length l is a sequence $v_0,H_1,v_1,H_2,v_2,\ldots,H_l,v_l$ with $v_i\in V(\mathcal{H})$ for $i=0,1,\ldots,l$ and $v_{i-1},v_i\in H_i\in\mathcal{E}(\mathcal{H})$ for $i=1,2,\ldots,l$. The distance $d_{\mathcal{H}}(u,v)$ of two vertices $u,v\in V(\mathcal{H})$ is the length of a shortest Berge path from u to v. The *ball centered at u and of radius l* consists of those vertices of $\mathcal{H}$ which are at distance at most l from u; it will be denoted by $B_l(u)$. We call $D\subset V(\mathcal{H})$ a *distance- l dominating set* of $\mathcal{H}$ if $\bigcup_{u\in D}B_l(u)=V(\mathcal{H})$. Equivalently we can say that $D\subset V(\mathcal{H})$ is a distance-l dominating set if and only if $D\cap B_l(v)\neq\emptyset$ for all $v\in V(\mathcal{H})$. Note that distance-1 dominating sets are the usual dominating sets.

The minimum size of a distance-l dominating set in a hypergraph $\mathcal{H}$ is the distance-l domination number $\gamma_d(\mathcal{H},l)$. Let further $n_d(k,\gamma,l)$ denote the minimum number of vertices that a k-uniform hypergraph $\mathcal{H}$ with no isolated vertices can contain if $\gamma_d(\mathcal{H},l)\geq\gamma$ holds. The next proposition shows that $n_d(k,\gamma,l)$ does not depend on l once $l\geq 2$ is supposed.

Proposition 1.2. *For any $k,l\geq 2$ and $\gamma\geq 1$ we have $n_d(k,\gamma,l)=k\gamma$, and the unique extremal hypergraph consists of γ pairwise disjoint edges.*

Proof. It is clear that the k-uniform hypergraph with just γ disjoint edges yields the upper bound $n_d(k,\gamma,l)\leq k\gamma$.

We prove the lower bound by induction on γ. The case $\gamma=1$ is trivial. So assume that $\gamma\geq 2$, and let $\mathcal{H}=(V(\mathcal{H}),\mathcal{E}(\mathcal{H}))$ be a k-uniform hypergraph with $\gamma_d(\mathcal{H},l)\geq\gamma$. Consider an arbitrary $v\in V(\mathcal{H})$. Any vertex in $N(B_{l-1}(v))$ is distance-l dominated by v, therefore the k-uniform hypergraph $\mathcal{H}'$ induced by the edge set $\{H\in\mathcal{E}(\mathcal{H}):H\cap B_{l-1}(v)=\emptyset\}$ covers all vertices of $\mathcal{H}$ not distance-l dominated by v. The assumption $\gamma_d(\mathcal{H},l)\geq\gamma$ implies $\gamma_d(\mathcal{H}',l)\geq\gamma-1$ and thus using that $|B_{l-1}(v)|\geq k$ for $l\geq 2$ and by induction we obtain

$$|V(\mathcal{H})|=|B_{l-1}(v)|+|V(\mathcal{H}')|\geq k+(\gamma-1)k=\gamma k.$$

Strict inequality holds whenever v has degree at least two. □

The problem becomes more interesting when disconnected hypergraphs get excluded. Hence, for $k\geq 2$ and $l,\gamma\geq 1$ let $n_{dc}(k,\gamma,l)$ denote the minimum number of vertices that a k-uniform **connected** hypergraph $\mathcal{H}$ must contain if it has $\gamma_d(\mathcal{H},l)\geq\gamma$.

To state our main result concerning $n_{dc}(k,\gamma,l)$ we need to define the following function:

$$f(k,\gamma,l):=\begin{cases}\dfrac{l}{2}k\gamma+\max\{k,\gamma\} & \text{if } l \text{ is even,}\\ \dfrac{l+1}{2}k\gamma & \text{if } l \text{ is odd.}\end{cases}$$

[2] The standard notation for s-tuple domination in the graph theory literature is $\gamma_{\times s}(G)$, but for the different variants of domination in this paper we try to use notations which are similar to each other in their form, this is why we put s in another position.

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